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Logistic Growth Model With Immigration Effect

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Abstract: **The paper studied a logistic growth model with immigration effect. The material was characterized by the modified logistic growth equation. The simplified population models usually start with four key variables including Death, Birth, Immigration and Emigration. It is understood that birth rates and carrying capacity of the population are affected by the presence of immigrants. In each special case, equilibrium points are used to compute birthrate and carrying capacity. The criteria of stability (stable, unstable and neutrally stable) with small perturbations around and all trajectories (Population vs Time) are established.**

Keywords: Logistic.Growth, growth model, population, timeseries, immigration effect.

I. Introduction

A Mathematical model is used to represent a collection of data in form of a diagrammatic representation such as bar chart, pie chart, histogram, box plot. The process of developing a mathematical model is termed as mathematical modelling. It involves the following processes. The mimicking of a real-world problem in mathematical terms; thus, the construction of mathematical model. By using mathematical model, the solution for analysis of data can be proved. The interpretation of the mathematical results in the context
of the original situation. Population level can be measured or verified by using mathematical methods [1].

II. MALTHUS GROWTH MODEL

Studies on single species population growth were initiated by Malthus 1798 and proposed a simple exponential growth model in his work ‘An Essay on the Principle of Population”. He developed a mathematical model hypothesizing that human populations have a constant natural growth rate [1].

If P(t)represent the population size at time t then Malthus Growth Model is

$dp/dt= ap$ (1)

The solution of the above equation is $p=p_0e^{at}$ (2)

where p_0 is the initial population.
a is the growth constant.

III. LOGISTIC GROWTH MODEL:

The Logistic growth model was proposed by Verhulst (1845). He incorporated in his model, the idea of carrying capacity. Thus, the population growth is not only depending on the population size, but also on the size from its upper limit, that is it carrying capacity[1]. He modified Malthus model

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to make a population size proportional to both the previous population and a new term

a-(bP(t))/a

Where a and b are vital coefficients of the population.

Hence, the modified equation is

dP/dt = aP (1-P/ (K)) (3)

Where K=a/ (b) is the carrying capacity

If immigration and emigration take place into the population from outside at a rate proportional to the

dx/dt =bx-dx+i-e=ax+k (4)

POPULATION GROWTH USING TIMESERIES

In logistic growth model, a population growth rate gets smaller as population size approaches a maximum imposed by limited resources in the environment, known as the carrying capacity. Logistic growth model produces an S-shaped curve. We can mathematically model logistic growth by considering the growth rate that depends on population size and how close it is to carrying capacity[2]. The logistic equation anticipates a limit or carrying capacity K on population growth is given as

dP/dt= aP(1-P/K), P_0=P (0) at t= 0 (5)

Solving for P(t) and simplifying further provide us with the solution

dP/dt = aP(1-P/K)

∫1/(P((K-P)/K)) dp = ∫adt (6)

= log P/(K-P) (7)

∫adt = at+c (8)

Sub (7) and (8) in (6)

Let A = (K-P_0)/P_0

Thus P(t) = (P_0 K)/(P_0+(K-P_0) e^(-at)) (9)

Plotting the population growth as a function of time shows P approaching K along a sigmoid (S-shaped) curve when the populations initial state $P_0$is belowK/2or aboveK/2.

Obtaining Equilibrium Points

We obtain the systems equilibrium pointsP^* by finding all values of P that satisfy

dP/dt = 0 (10)

aP^* (1-P^*/K) = 0 (11)

P^*=0 or P^*=K

That is P(t) approaches to K

At P^* = 0 P(t) - unstable

P^* = K P(t) - stable

Let P(t) = P^*+ε(t)

Where ∑(t) a small perturbation in the neighborhood of an equilibrium point denoted is P^*

ε = P-P^*

dε/dt = d/dt (P-P^*) (12)

By Appling Taylor’s expansion

f(P^*+ε) = f (P^*) +ε df/dP

For equilibrium

P^*=0, dε/dt=(a-2aP/K) ε= aε

P=K, dε/dt=(a-2aP/K) ε=-aε

dε/dt=±aε (13)

Logistic Growth Equation with Immigration

Let f(P) be the immigration function. Then the modified logistic growth equation is given by

dP/dt = aP(1-P/K) +f(P) (14)

The system’s equilibrium point P^*can be obtained by finding all the values of P that satisfy the equation

(dP^*)/dt = aP^* (1-P^*/K) +f (P^*) = 0 (15)

To analyze the stability of equilibrium point, consider a small perturbation around equilibrium point, P(t)=P^*+ε(t), where ε is a first order small quantity. Using equation, we get after neglecting the higher power of ε the differential equation for ε

dε/dt = [bK-2bP^*+f^' (P^*)] ε (16)

SPECIAL CASES FOR THE IMMIGRATION FUNCTION F(P)

CASE A: f(P)=c (constant)

The modified Logistic equation is

= aP-(aP^2)/K+c (17)

The equilibrium points are given by

P^* = (bK±√((bK)^2+4bc))/2b (18)

The linearized equation forε is given by

dε/dt = [bK-2bP^*+ f^' (P^*)] ε (19)

On integrating we get

ε = $ε_0$ e^(-t) (20)

Where=√((bK)^2+4bc)>0, as t→∞, ε→0

The solution of the logistic equation is given by

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From equation (17)
$dP/dt=(-a)/k\ [P^2-KP-c/a]$

On integrating,
$(-K)/a \int dP/(P^2-KP-c/a) = dt$ (21)

Consider, $(-K)/a \int dP/(P^2-KP-c/a)$
$P = K/2+v.\ (1+c_1 e^{(-xt)})/ (1-c_1 e^{(-xt)})$ (22)
At $t_0=0$, $P = P_0$
$C_1=(2b(P_0-v)-a)/(2b(P_0+v)-a) = u$ (say)
Substituting C_1 in (22), we get
$P = K/2+v.(1+ue^{(-2bvt)})/(1-ue^{(-2bvt)})$ (23)
Where
$u=(2b(P_0-v)-a)/(2b(P_0+v)-a)$ and
$v=((bk)^2+4cb)/2b$

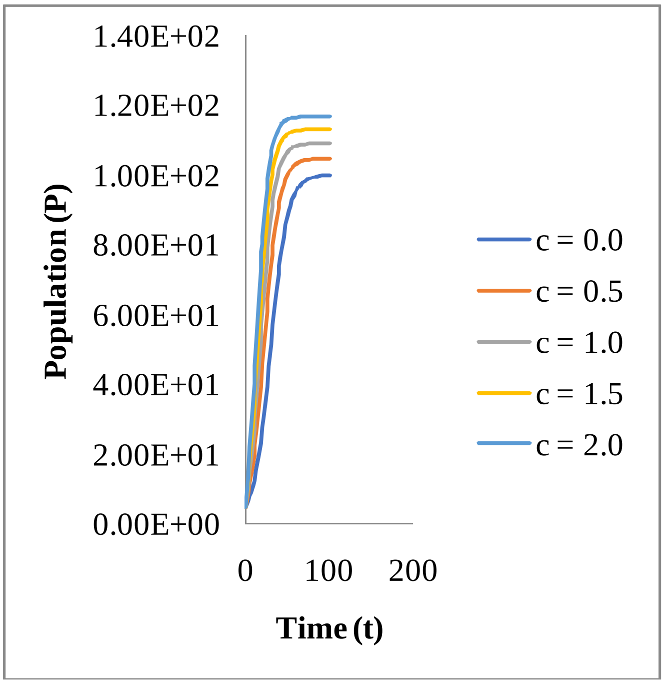

Fig 1. Population vs Time (P_0<K/2, P_0=5, K=100, b=0.001)

Fig 1.Shows the relation between the population and time. As time increases the population also increases for different values of c.

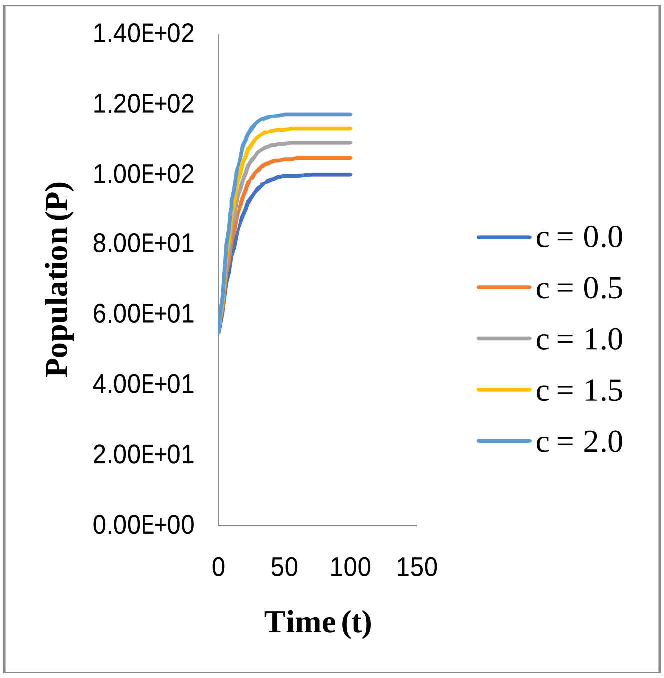

Fig 2. Population vsTime(P_0>K/2, P_0=55, K=100, b=0.001)

Fig 2.Presents the relation between the population and time. As time increases the population also increases for different values of c. It is clear that increase in carrying capacity increases the population.

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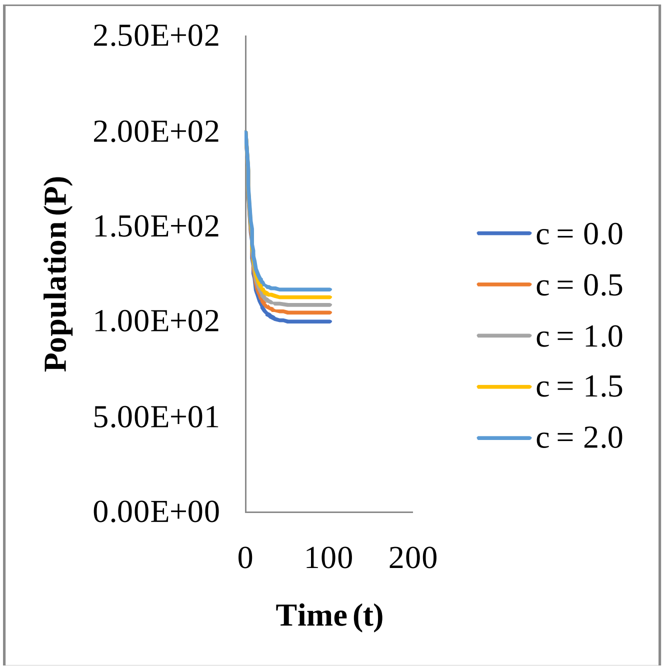

Fig 3. Population vs Time (P_0>K, P_0=200, K=100, b=0.001)

Fig 3.Exhibits the relation between the population and time. With the increase in time the population decreases increase for different values of c.

CASE B: f(P) ∝P that is f(P) = cP
The modified Logistic growth equation is given by

dP/dt = bP[(K+c/b)-P] (24)
The equilibrium points are given by
dε/ε= adt (25)
Integrating we get,
ε = ε_0 e^at,a>0 as t→∞, ε→∞ (26)
∴P_1^* = 0 is unstable
For P_2^* = (a+c)/bthe linearized equation is given as
dε/dt = [bK – 2b((a+c)/b)+ c]ε
Here, the carrying capacity is increased from K to K+c/b

The solution is given by
P = ((K+c/b) P_0)/(P_0+(K+c/b-P_0) e^(-Kbt))
(27)

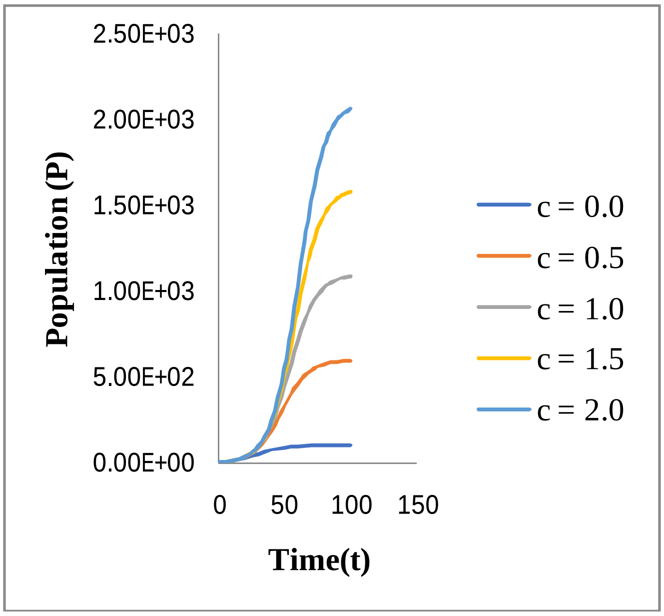

Fig 4. Population vs Time (P_0<K/2,P_0=55, K=100, b=0.001)

Fig 4. Shows the relation between the population and time when P_0<K/2. As time increases the population also increases for different values of c. with the increase in c the population also increases.

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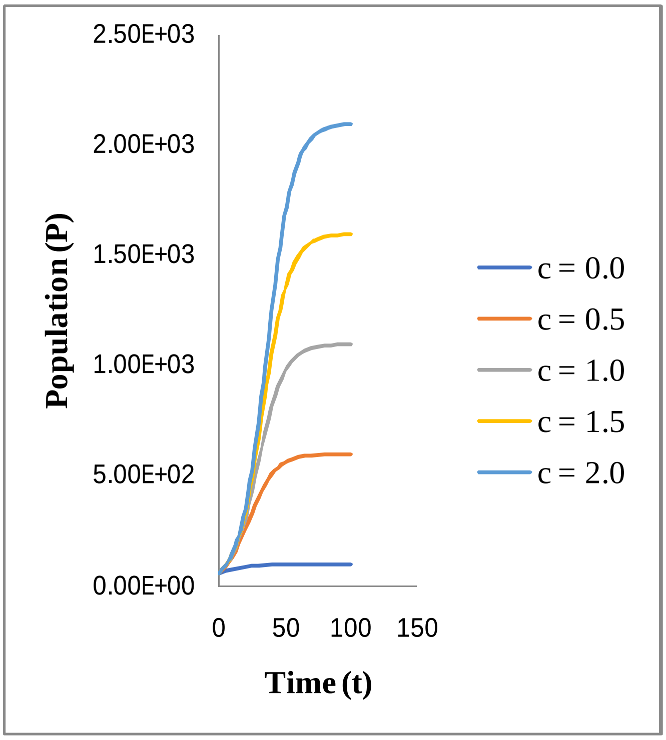

Fig 5. Populationvs Time (P_0>K/2, P_0=55, K=100, b=0.001)

Fig 5. Exhibits the relation between the population and time when P_0>K/2. As time increases the population also gets increased for different values of c. with the increase in c the population also increases.

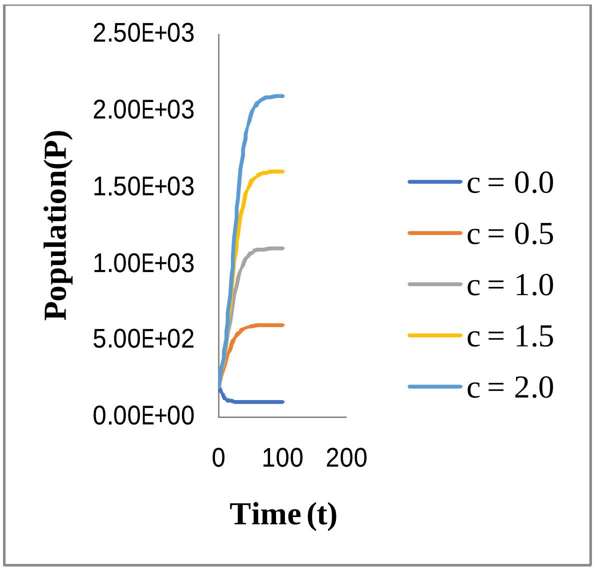

Fig 6. Population vs Time (P_0>K, P_0=200, K=100, b=0.001)

Fig 6. Shows the relation between the population and time when P_0>K. As time increases the population also increases for different values of c. with the increase in c the population also increases.

CASE C: f(P) ∝P^2 that is f(P) = cP^2
The modified Logistic growth equation is given by
$= bP [K – (1 - c/b) P]$ (28)
To obtain equilibrium point
P_1^*=0 P_2^*=K/(1-c/b) =a/(b-c)
For P_1^*=0, the linearized equation for ε is
∴ε = ε_0e^at,a>0 as t→∞, ε→∞
∴P_1^*= 0 is unstable.
For P_2^* = a/(b-c), the linearized equation is given as
dε/dt = (a- 2a) ε (29)
∴ε = ε_0e^at, as t→∞, ε→0 (30)
∴P_2^* = a/(b-c) is stable
Here, the carrying capacity is increased from K to a/(b-c)
The solution is given by

P = ((bK/(b-c)) P_0)/(P_0+(bK/(b-c)-P_0) e^(-Kbt)) (31)

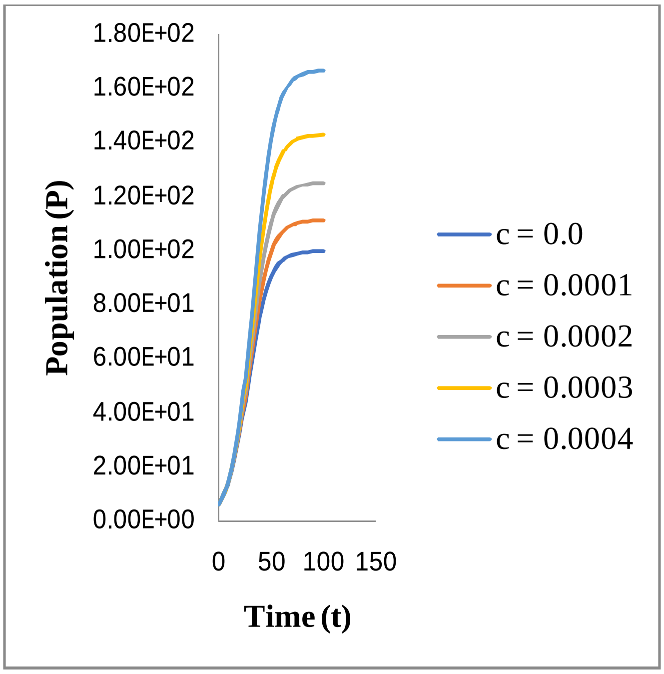

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Fig7. Populationvs Time (P_0>K/2,P_0=6,K=100,b=0.001)

Fig 7. Shows the relation between the population and time when P_0<K/2. As time increases the population also increases for different values of c. with the increase in c the population also increases however small ‘c’ be.

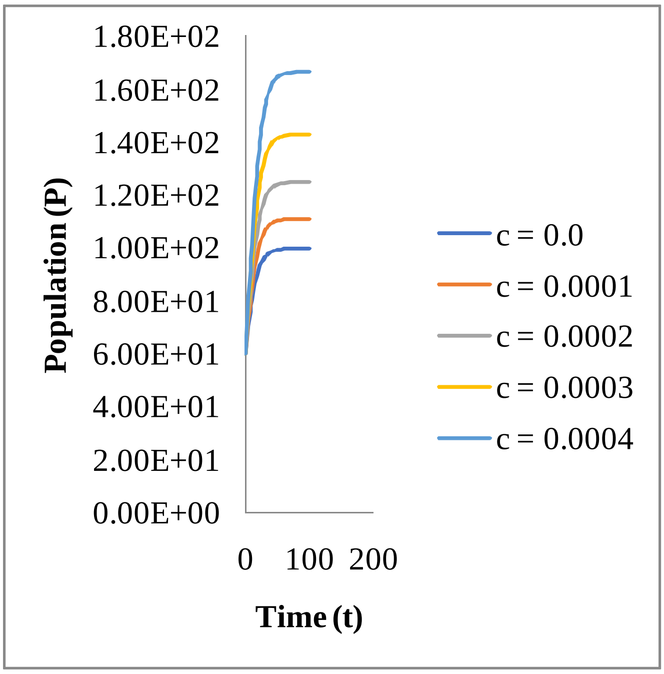

Fig8. Population vs Time (P_0>K/2, P_0=55, K=100, b=0.001)

Fig 8. Depicts the relation between the population and time when P_0>K/2. As time increases the population also increases for different values of c. with the increase in c the population also increases.

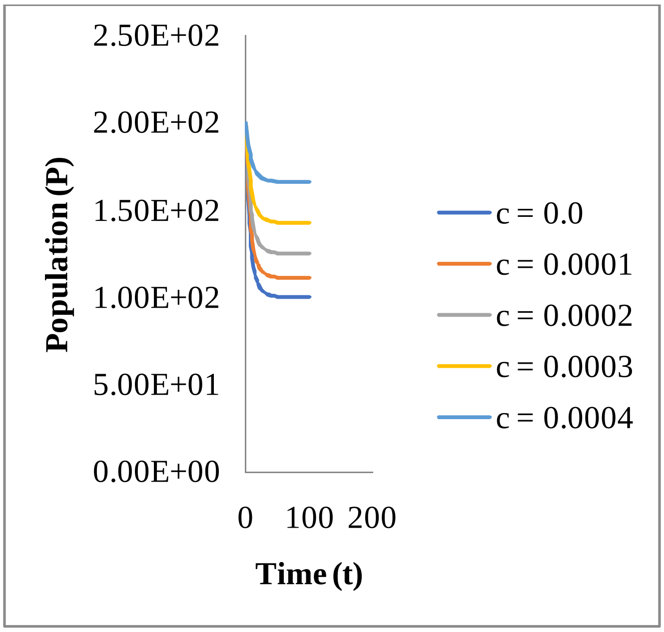

Fig9. Population vs Time (P_0>K/2, P_0=55, K=100, b=0.001)

Fig 9. Shows the relation between the population and time when P_0>K. As time increases the population decreases for different values of c. with the increase in c the population decreases for small values of ‘c’.

CONCLUSION

The analysis of Logistic growth population model has quantitative measure of systems stability. We considered a small perturbation in the neighborhood of equilibrium pointP^*, and obtained the results P^*=0unstable and P^*=K is stable. Secondly addedan immigration characteristic f(P) into classical logistic growth equation and analyzed the stability of equilibrium and analyze the stability of equilibrium states of logistic equation in three special cases. In each of the cases the effective birth rate, carrying capacity are computed, equilibrium points are obtained and stability criteria is established. All the trajectories show that the carrying capacity increases with a small variation of ‘c’. It is concluded that the birthrate and carrying capacity of population are affected by the immigrants.

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