ISSN:0971-1260
Vol-22-Issue-10-November-2019

Inference under progressive interval censoring for Rayleigh competing risk failure model.

Neha K. Gadhvi
Department of Statistics
Ahmedabad University
Ahmedabad-380009
Email: nehagadhvi@gmail.com

ABSTRACT

In many cases failure of individuals may be due to more than one case or factor, for such cases competing risk failure model is used instead of simple failure model. In this paper Rayleigh life time model is used as a model for competing risk failure analysis. The cases of failure are assumed independent for inference type-I interval censoring scheme is used. The point estimation and interval estimation are done based on maximum likelihood as well as Bootstrap method. The estimation is done for the parameters and survival function of the competing risk failure method. Non parametric estimation of survival function is also considered. A simulation study is done to study the behaviour of the estimator.

1. Introduction:

Failure data analysis is used in engineering medicine and many other fields. The failure time data arises as time to AIDS for HIV individuals, the time to death for cancer patients, life of electrical or electronic components etc.
The failure of an item may be due to one or more than one cases. When failure is observed due to multiple cases the analysis of such failure time data becomes more patient where cause of death was recorded as "Cancer" or "other". Other example is death of patients may be due to Kidney dieses or heart attack, such types of competing risk failure model is studied by many authors like peck Boardman and Kendell(1970), Hoel(1972), Patel and Gajjar(1992).
Recently Gadhvi and Bhimani (2015, 2016) have considered competing risk failure model for geometric as well as exponential life time models.
In competing risk failure the causes of failure may be assumed independent or dependent. In most of the situations independent causes of failure is assumed for the data analysis, since there is some concern about the identifiability of the underlying model even though the assumption of dependent may be more realistic.

Suppose that an unit fails due to one of the K different independent causes. When the unit begins operation each failure cause simultaneously generate a random life. Thus in effect K lifetimes $X_1, X_2, \ldots X_K$ is realized. If X is a lifetime of the unit then we have,

$$X = \min\{ X_1, X_2, \ldots X_K\} = X_{(1)}$$

The cumulative distribution function(cdf) of X is given by,

$$F_X(X) = 1 - \prod_{i=1}^{K} [1 - F_{Xi}(X)]$$

ISSN:0971-1260
Vol-22-Issue-10-November-2019

Here each failure mode can have distribution that not all failure distribution need be alike, but all the k modes operate independently of each other. Such a model is known as competing risk failure model.
May life time distribution like exponential, Weibull, Pareto, power function etc. are used as life time model. Rayleigh life time distribution is used when the failure rate increase with time. Rayleigh distribution was inverted by Lord Rayleigh(1880), during a study of acoustical problems. Rayleigh distribution possesses many interesting properties. It is widely useful in communication engineering (Dyer and Whisenand(1973)), electro vacuum devices Polovko (1968)) and in clinical studies dealing with cancer patients(Bhattacharya and Tyagi(1990)). This distribution was successfully employed as a radio wave power distribution (Siddiqui(1962), Shah and Patel(2008)) have considered Rayleigh distribution as a life time model.

2. Competing Risk Failure model:

Suppose that the device or individual exhibits two models of failure and each failure mode simultaneously generate a random lifetime X_1 and X_2 respectively. i.e. X_1 is the time of failure of an item due to cause i then the probability density function(pdf) of X_i, in case of Rayleigh distribution is given by,

$$f_i(x) = \frac{x}{\theta_i^2} e^{-x^2/2\theta_i^2}, x > 0, \theta_i > 0, i = 1, 2. \qquad ..(2.1)$$

And corresponding cdf is,

$$F_i(x) = 1 - e^{-x^2/2\theta_i^2}, i = 1, 2. \qquad ..(2.2)$$

Let T be the failure time of the item regardless of cause then T = min(X_1, X_2). The pdf of T is given by,

$$H_T(t) = P(T \le t)$$

$$= 1 - \prod_{i=1}^{2} \left[1 - F_i(t)\right]$$

$$= 1 - \left(e^{-t^2/2\theta_1^2}\right)\left(e^{-t^2/2\theta_2^2}\right)$$

$$= 1 - e^{-t^2/2\theta^2} \qquad ..(2.3)$$

And its pdf is given by,

$$h_T(t) = \frac{dH_T(t)}{dt} = \frac{t}{\theta^2} e^{-t^2/2\theta^2}, \text{ where } \frac{1}{\theta^2} = \frac{1}{\theta_1^2} + \frac{1}{\theta_2^2} \qquad ..(2.4)$$

ISSN:0971-1260
Vol-22-Issue-10-November-2019

Which is also a Rayleigh distribution.

Let $P_i(t)$ be the probability that an item failed by cause $i$at the time and it must not failed by the cause $i' \neq i$ upto time t ; $i, i' = 1,2.$

Then $P_i(t) = f_i(t)[1 - F_{i'}(t)]$; $i, i' \neq 1,2.$

Hence probability of failure of an item at time t is given by,

$$P(t) = P_1(t) + P_2(t)$$

$$= f_1(t)\left[1 - F_2(t)\right] + f_2(t)\left[1 - F_1(t)\right]$$

$$= \frac{t}{\theta_1^2} e^{-t^2/2\theta_1^2} . e^{-t^2/2\theta_2^2} + \frac{t}{\theta_2^2} e^{-t^2/2\theta_2^2} . e^{-t^2/2\theta_1^2}$$

$$= \frac{t}{\theta_1^2} e^{-t^2/2\theta^2} + \frac{t}{\theta_2^2} e^{-t^2/2\theta^2}$$

$$= g_1(t) + g_2(t) \qquad ..(2.5)$$

Where, $\frac{1}{\theta^2} = \frac{1}{\theta_1^2} + \frac{1}{\theta_2^2}$ and $g_i(t) = \frac{t}{\theta_i^2} e^{-t^2/2\theta^2}, i = 1, 2.$ ..(2.6)

the model in (2.5) is called Rayleigh competing risk failure model, which is same as(2.4).

3. Interval Censoring:

Suppose that the failure observed between the given time interval (a,b), a<b. here experimental units. Here experimental units are not monitored continuously. In such situation interval censored data arise.

In medical studies such types of data arise when patients are only monitored regular interval (weekly or quarterly check up). Many authors have discussed application of interval censoring in clinical, medical, bio-medical and engineering studies like Odell et. al (1992), Sumuelson and Kongerud(1994), Scallan(1999), Rao(1998), Aggarwala(2001) etc. Such censoring scheme is also known as group censoring scheme. Patel and Gajjar(1992), Patel and Gajjar(2007) have considered maximum likelihood estimation under progressive grouped censored samples in case of compound geometric lifetime models with different parameters at different stages of censoring.

Under progressive type I interval censored sample. The likelihood function is given by,

$$L(\theta_1, \theta_2) \propto \prod_{i=1}^{m}\left[G_1(T_i) - G_2(T_{i-1})\right]^{x_{1i}} \prod_{i=1}^{m}\left[G_2(T_i) - G_2(T_{i-1})\right]^{x_{2i}} \prod_{i=1}^{m}\left[1 - H(T_i)\right]^{r_i} \qquad ..(3.1)$$

ISSN:0971-1260
Vol-22-Issue-10-November-2019

Where r_i denotes the fixed numbering items removed at the end of the ith time interval (T_{i-1} , T_i), i.e. at time T_i , i=1,2,….,m. here m=the fixed pre-determined number of time intervals; $T_0=0$, n= total number of items put on a test, i.e. sample size.
Let x_{ji} denotes the number of failure during the time interval [T_{i-1} , T_i], i=1,2,…,m. due to jth cause of failure, j=1,2. and $r_m = n - \sum_{i=1}^{m} x_{1i} - \sum_{i=1}^{m} x_{2i} - \sum_{i=1}^{m-1} r_i$

Here, $$G_1(T_i) = P(X_1 \le T_i)$$

$$= \int_0^{T_i} g_1(t)\,dt = \int_0^{T_i} \frac{t}{\theta_1^2} e^{-t^2/2\theta^2}\,dt$$

$$= \frac{\theta^2}{\theta_1^2}\left(1 - e^{-T_i^2/2\theta^2}\right) \qquad ..(3.2)$$

Similarly, $$G_2(T_i) = P(X_2 \le T_i)$$

$$= \int_0^{T_i} g_2(t)\,dt = \int_0^{T_i} \frac{t}{\theta_2^2} e^{-t^2/2\theta^2}\,dt$$

$$= \frac{\theta^2}{\theta_2^2}\left(1 - e^{-T_i^2/2\theta^2}\right) \qquad ..(3.3)$$

Using(2.3),(3.2) and (3.3) in (3.1) we get the likelihood function under progressive type-I interval censoring with two independent causes of failures for Rayleigh lifetime model as,

$$L = L(\theta_1, \theta_2) = \text{constant}.\prod_{i=1}^{m}\left\{\frac{\theta^2}{\theta_1^2}\left(e^{-T_{i-1}^2/2\theta^2} - e^{-T_i^2/2\theta^2}\right)\right\}^{x_{1i}} .\prod_{i=1}^{m}\left\{\frac{\theta^2}{\theta_2^2}\left(e^{-T_{i-1}^2/2\theta^2} - e^{-T_i^2/2\theta^2}\right)\right\}^{x_{2i}} .\prod_{i=1}^{m}\left(e^{-T_i^2/2\theta^2}\right)^{r_i}$$

$$= \text{constant}.\frac{\left(\theta^2\right)^{\sum_1^m x_i}}{\left(\theta_1^2\right)^{\sum_1^m x_{1i}}.\left(\theta_2^2\right)^{\sum_1^m x_{2i}}}.e^{-\frac{1}{2\theta^2}\sum_{i=1}^{m}\left\{T_{i-1}^2 x_i + T_i^2 r_i\right\}}.\prod_{i=1}^{m}\left\{1 - e^{-S_i/2\theta^2}\right\}^{x_i} \qquad ..(3.4)$$

Where, $x_i = x_{1i} + x_{2i}$, $\mathrm{S}_i = T_i^2 - T_{i-1}^2$, $i = 1,...,m.$
Hence,

$$\log L = 2\sum_i^m x_{2i}\log\theta_1 + 2\sum_i^m x_{1i}\log\theta_2 - \sum_i^m x_i \log\left(\theta_1^2 + \theta_2^2\right) - \frac{A_1}{2}\left(\frac{1}{\theta_1^2} + \frac{1}{\theta_2^2}\right) + \sum_1^m x_i \log\left(1 - e^{-\frac{S_i}{2}\left(\frac{1}{\theta_1^2} + \frac{1}{\theta_2^2}\right)}\right)$$

Where, $A_1 = \sum_{i=1}^{m}\left\{T_{i-1}^2 x_i + T_i^2 r_i\right\}$ ..(3.5)

Differentiating logL with respect to θ_1 and θ_2 & comparing them with zero, we get

ISSN:0971-1260
Vol-22-Issue-10-November-2019

$$\frac{\partial \log L}{\partial \theta_1} = \frac{2\sum_1^m x_{2i}}{\theta_1} - \frac{2\sum_1^m x_i\theta_1}{\theta_1^2+\theta_2^2} + \frac{A_1}{\theta_1^3} - \sum_1^m x_i \left\{ \frac{\frac{S_i}{\theta_1^3} e^{-S_i/2\theta^2}}{1-e^{-S_i/2\theta^2}} \right\} = 0 \qquad ..(3.6)$$

$$\frac{\partial \log L}{\partial \theta_2} = \frac{2\sum_1^m x_{1i}}{\theta_2} - \frac{2\sum_1^m x_i\theta_2}{\theta_1^2+\theta_2^2} + \frac{A_1}{\theta_2^3} - \sum_1^m x_i \left\{ \frac{\frac{S_i}{\theta_2^3} e^{-S_i/2\theta^2}}{1-e^{-S_i/2\theta^2}} \right\} = 0 \qquad ..(3.7)$$

Solving (3.6) & (3.7) we get,

$$\theta_1 = \theta_2 \sqrt{\frac{\sum x_{2i}}{\sum x_{1i}}} \qquad ..(3.8)$$

Putting(3.8) into(3.6), we get the equation in single parameter $\theta_2$as,

$$\theta_2^2 = \frac{2\sum_1^m x_{1i} a_1^2 \theta^2}{\sum_1^m x_{2i}\theta^2 + A_1 - \sum_{i=1}^m \left\{ \frac{x_i S_i e^{-S_i/2\theta^2}}{1-e^{-S_i/2\theta^2}} \right\}} \qquad ..(3.9)$$

$$\theta_2 = \psi(\theta_2) \qquad ..(3.10)$$

where $\Psi(\theta_2)$ will be a function of only parameter θ_2, which is a square root of the RHS. of (3.9).
Solving the equation (3.10) for $\theta_2$by any method of interaction, we get $\widehat{\theta_2}$, mle of θ_2. And again substitution $\widehat{\theta_2}$ in place of θ_2 in (3.8). we get $\widehat{\theta_1}$mle of θ_1.

The survival function at time t_0 is given by,

$$S = S(t_0) = 1 - H(t_0) = e^{-t_0^2/2\theta^2} \qquad ..(3.11)$$

Replacing $\theta_1$and $\theta_2$by their mle $\widehat{\theta_1}$ and $\widehat{\theta_2}$ in (3.11) we get MLE of survival function,

$$\hat{S}(t_0) = e^{-t_0^2/2\hat{\theta}^2} \qquad ..(3.12)$$

where,

$$\frac{1}{\hat{\theta}^2} = \frac{1}{\hat{\theta}_1^2} + \frac{1}{\hat{\theta}_2^2}$$

4. Standard errors of the estimators:

ISSN:0971-1260
Vol-22-Issue-10-November-2019

The asymptotic variance and covariance matrix Σ of the MLE for the parameters θ_1 and θ_2 are given by elements of the inverse of the Fisher information matrix,

$$I_{ij} = E\left(-\frac{\partial^2 \log L}{\partial\theta_i \partial\theta_j}\right); \qquad i, j = 1, 2$$

That is,

$$\Sigma = \begin{pmatrix} V\left(\hat{\theta}_1\right) & Cov\left(\hat{\theta}_1, \hat{\theta}_2\right) \\ Cov\left(\hat{\theta}_1, \hat{\theta}_2\right) & V\left(\hat{\theta}_2\right) \end{pmatrix} = \begin{bmatrix} -E\left(\frac{\partial^2 \log L}{\partial\theta_1^2}\right) & -E\left(\frac{\partial^2 \log L}{\partial\theta_1 \partial\theta_2}\right) \\ -E\left(\frac{\partial^2 \log L}{\partial\theta_1 \partial\theta_2}\right) & -E\left(\frac{\partial^2 \log L}{\partial\theta_2^2}\right) \end{bmatrix} \qquad ..(4.1)$$

where,

$$\frac{\partial^2 \log L}{\partial\theta_1^2} = \frac{-2\sum_1^m x_{2i}}{\theta_1^2} - \frac{2\sum_1^m x_i\left(\theta_2^2 - \theta_1^2\right)}{\left(\theta_1^2 + \theta_2^2\right)^2} - \frac{3A_1}{\theta_1^4} + \frac{3}{\theta_1^4}\sum_1^m \left\{\frac{x_i S_i e^{-S_i/2\theta^2}}{1 - e^{-S_i/2\theta^2}}\right\} - \frac{1}{\theta_1^6}\sum_1^m \left\{\frac{x_i S_i e^{-S_i/2\theta^2}}{\left(1 - e^{-S_i/2\theta^2}\right)^2}\right\} \qquad ..(4.2)$$

$$\frac{\partial^2 \log L}{\partial\theta_2^2} = \frac{-2\sum_1^m x_{2i}}{\theta_2^2} - \frac{2\sum_1^m x_i\left(\theta_1^2 - \theta_2^2\right)}{\left(\theta_1^2 + \theta_2^2\right)^2} - \frac{3A_1}{\theta_2^4} + \frac{3}{\theta_2^4}\sum_1^m \left\{\frac{x_i S_i e^{-S_i/2\theta^2}}{1 - e^{-S_i/2\theta^2}}\right\} - \frac{1}{\theta_2^6}\sum_1^m \left\{\frac{x_i S_i e^{-S_i/2\theta^2}}{\left(1 - e^{-S_i/2\theta^2}\right)^2}\right\} \qquad ..(4.3)$$

$$\frac{\partial^2 \log L}{\partial\theta_1 \partial\theta_2} = \frac{4\theta_1\theta_2\sum_1^m x_i}{\left(\theta_1^2 + \theta_2^2\right)^2} - \frac{\sum_1^m x_i S_i^2 \left\{\frac{e^{-S_i/2\theta^2}}{\left(1 - e^{-S_i/2\theta^2}\right)^2}\right\}}{\theta_1^3\theta_2^3} \qquad ..(4.4)$$

The asymptotic variance of MLE of survival function is given by,

$$V\left(\hat{S}(t_0)\right) = \left(\frac{\partial S}{\partial\theta_1}\right)^2 V\left(\hat{\theta}_1\right) + \left(\frac{\partial S}{\partial\theta_2}\right)^2 V\left(\hat{\theta}_2\right) + 2\left(\frac{\partial S}{\partial\theta_1}\right)\left(\frac{\partial S}{\partial\theta_2}\right)\mathrm{cov}\left(\hat{\theta}_1, \hat{\theta}_2\right) \qquad ..(4.5)$$

ISSN:0971-1260
Vol-22-Issue-10-November-2019

Where, $\frac{\partial S}{\partial \theta_1} = \frac{t_0^2}{\theta_1^3} e^{-t_0^2 / 2\theta^2}$ and $\frac{\partial S}{\partial \theta_2} = \frac{t_0^2}{\theta_2^3} e^{-t_0^2 / 2\theta^2}$

5. Simulation Algorithm:

A study of properties of maximum likelihood estimators based on progressively Type-I interval censored samples involves simulation. A short algorithm for simulating a random sample of size n put on a life test at time 0 is given below. Here we use the following properties of the progressive interval censoring.
$X_{11} \sim B\left(n, G_1(T_1)\right)$ and $X_{21} \sim B\left(n, G_2(T_1)\right)$

Where $G_1(T_1)$ and $G_2(T_1)$ are defined in (3.3) and (3.4) respectively and for i = 2,3,…,m

$$X_{ji} / X_{ji-1}, X_{ji-2}, X_{ji-3,\ldots,} X_{j1}, R_{i-1}, R_{i-2}, \ldots, R_1 \sim B\left(n - \sum_{s=1}^{i-1}\left(X_{1s} + X_{2s} + R_s\right); \frac{G_j(T_i) - G_j(T_{i-1})}{1 - \sum_{s=1}^{i-1}\left\{G_j(T_i) - G_j(T_{i-1})\right\}} \right)$$

$$= B\left(n - \sum_{s=1}^{i-1}\left(X_{1s} + X_{2s} + R_s\right); \frac{G_j(T_i) - G_j(T_{i-1})}{1 - G_j(T_{i-1})} \right); j = 1, 2.$$

Here $B(n,p)$ denotes the binomial distribution with parameters $n$and p; $0 < p < 1$.
On the basis of the algorithm given in Aggarwala (2001) for simulating a sample under progressive Type-I interval censoring scheme Gadhvi and Bhimani (2015) have modify the simulation scheme for competing risk failure model. The main steps of the algorithm are given below:

1. Set $i = 0, X_{1sum} = 0, X_{2sum} = 0, R_{sum} = 0$
2. Next i
3. If $i = m + 1$, exist the algorithm.
4. Generate X_{11} and X_{21} as binomial random variables with parameters $\left(n, G_1(T_1)\right)$ and $\left(n, G_2(T_1)\right)$ respectively.
5. Generate X_{1i} and X_{2i} as binomial variables with parameter

$$\left(n - X_{1sum} - X_{2sum} - R_{sum}, \frac{G_1(T_i) - G_1(T_{i-1})}{1 - G_1(T_{i-1})} \right)$$

and

$$\left(n - X_{1sum} - X_{2sum} - R_{sum}, \frac{G_2(T_i) - G_2(T_{i-1})}{1 - G_2(T_{i-1})} \right)$$ respectively.

ISSN:0971-1260
Vol-22-Issue-10-November-2019

6. Calculate $R_i^{obs} = Floor\left[p_i \left(n - X_{1sum} - X_{2sum} - R_{sum} \right) \right]$ or $\min\left(R_i, n - X_{1sum} - X_{2sum} - R_{sum} - X_i \right)$.
7. Set $X_{1sum} = X_{1sum} + X_{1i}, X_{2sum} = X_{2sum} + X_{2i}, R_{sum} = R_{sum} + R_i^{obs}$
   This algorithm generates m binomial random variables. Here either the values $p_1, p_2, \dots, p_{m-1}$ or proposed values of $R_1, R_2, \dots, R_m$ are fixed in advance by the experimenter. Here $p_m = 1$and $R_m = n - \sum_1^m X_{1i} - \sum_1^m X_{2i} - \sum_1^{m-1} R_i$.

6. Confidence Interval Estimation:

A. Asymptotic Confidence Interval:

Using the asymptotic normality property of maximum likelihood estimator, confidence interval for MLE can be obtained for parameters θ_1, $\theta_2$as and survival function $S(t_0)$ as follows:
$(1-\propto)100\%$ asymptotic confidence interval for $\hat{\theta}_i$ becomes

$$\hat{\theta}_i \pm Z_{\alpha/2} \sqrt{V\left(\hat{\theta}_i\right)}, i = 1, 2$$

And for $S(t_0)$:

$$\hat{S}\left(t_0\right) \pm Z_{\alpha/2} \sqrt{V\left(\hat{S}\left(t_0\right)\right)}$$

Where $V(\hat{\theta}_i)$ and $V\left(\hat{S}(t_0)\right)$ can be obtained from equations (3.14) & (3.18) and $Z\alpha/_2$ is$\left(\alpha/_2\right)^{th}$ percentile of standard normal distribution.

B. Bootstrap Confidence Interval:

B-I: Percentile Bootstrap Method:

1. From the original data X compute the ML estimates of the parameters say $\hat{\theta}_1$ and $\hat{\theta}_2$ from (3.12) and (3.10).
2. Use $\hat{\theta}_1$ and $\hat{\theta}_2$ to generate a bootstrap sample $\underline{X}^*$with the same values of T_i, r_i and m, $i = 1, 2, \dots, m$, using the algorithm given in section 5.
3. As discussed in step 1, based on $\underline{X}^*$ compute the bootstrap sample estimator of θ_1, θ_2 and $S(t_0)$, say $\hat{\theta}_1^*, \hat{\theta}_2^*$ and $\hat{S}^*\left(t_0\right)$.
4. Repeat steps 2 and 3, S times representing S bootstrap MLE's of $\left(\theta_1, \theta_2, S(t_0)\right)$ based on S different bootstrap samples.
5. Arrange all $\hat{\theta}_1^*, \hat{\theta}_2^*$ and $\hat{S}^*\left(t_0\right)$ in an ascending order to obtain bootstrap sample $\left(\phi_l^{[1]}, \phi_l^{[2]}, \dots, \phi_l^{[S]}\right), l = 1, 2, 3.$where $\phi_1 = \hat{\theta}_1^*, \phi_2 = \hat{\theta}_2^*$ and $\phi_3 = \hat{S}^*\left(t_0\right)$.
6. Let $G\left(Z\right) = P\left(\phi_l \le Z\right)$be the cumulative distribution function of ϕ_l. Define $\phi_{lboot} = G^{-1}\left(Z\right)$for given Z. The approximate bootstrap $(1-\alpha)100\%$ confidence interval of ϕ_l is given by $\left[\phi_{lboot}\left(\alpha/2\right), \phi_{lboot}\left(1-\alpha/2\right)\right]$.

ISSN:0971-1260
Vol-22-Issue-10-November-2019

B-II: Bootstrap-t method:

1. From the original data $\underline{x}$ compute the MLE of the parameters say, $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{S}(t_0)$.
2. Use $\hat{\theta}_1 and \hat{\theta}_2$ to generate bootstrap sample $\underline{X}^*$with the same values of T_i, r_i and m, $i = 1, 2, \ldots, m$, using the algorithm given in B-I.
3. Based on $\underline{X}^*$ compute the bootstrap sample estimates of θ_1, θ_2 and $S(t_0)$, say $\hat{\theta}_1^*, \hat{\theta}_2^*$ and $\hat{S}^*(t_0)$.
4. Compute the following statistics:
$$T_1^* = \frac{\sqrt{n}\left(\hat{\theta}_1^* - \hat{\theta}_1\right)}{\sqrt{V\left(\hat{\theta}_1^*\right)}}, T_2^* = \frac{\sqrt{n}\left(\hat{\theta}_2^* - \hat{\theta}_2\right)}{\sqrt{V\left(\hat{\theta}_2^*\right)}}, T_3^* = \frac{\sqrt{n}\left(\hat{S}^*(t_0) - \hat{S}(t_0)\right)}{\sqrt{V\left(\hat{S}^*(t_0)\right)}}.$$
where $V(\,)$ can be obtained from (3.14).
5. Repeat step 3 and 4 S boot times.
6. From the values of $T_i^*, i = 1,2,3$ obtained in step 4, determine the upper and lower bounds of the $100(1-\alpha)\%$ confidence interval of the parameters and survival function as follows:
Let $H(x) = P\left(T_i^* \le x\right), i = 1,2,3$ be the cdf of T_i^*. For a given x define;
$\left(\hat{\theta}_{1Boot-t}\left(\alpha/2\right), \hat{\theta}_{1Boot-t}\left(1-\alpha/2\right)\right)$. Similarly we can define other parameters.

7. Non Parametric Estimation of Survival Function.

Kaplan Meier estimate of survival function $S(t_0)$ can be obtained according to Miller et al (1981 pg. 47,51). The estimate and estimate of the variance can be obtained as follow:

$$\hat{S}(t_0) = \prod_{Y_{(i)} \le t_0} \left(\frac{n-i}{n-i+1}\right)^{\delta_{(i)}} \qquad ..(7.1)$$

and
$$Asy\hat{V}\left(\hat{S}(t_0)\right) = \hat{S}^2(t_0) \sum_{y_{(i)} \le t_0} \frac{\delta_{(i)}}{(n-i)(n-i+1)} \qquad ..(7.2)$$

Using the results (7.1) & (7.2), the $(1-\propto)100\%$ asymptotic confidence interval for $\hat{S}(t_0)$ can be obtained

$$\hat{S}(t_0) \pm Z_{\alpha/2} \sqrt{Asy\hat{V}\left(\hat{S}(t_0)\right)} \qquad ..(7.3)$$

8. Simulated Example:

In this section we have simulated the data considering the parametric values $\theta_1 = 12$, $\theta_2 = 8$, and time t = 6 for reliability R(t). A sample of size 300 is generated for 10 stages of censoring with two causes of failures using the using the algorithm given in Section 5.

Time intervals are assumed of equal length 2 as 0 – 2, 2 – 4, ... and up to 18 – 20.

ISSN:0971-1260
Vol-22-Issue-10-November-2019

The simulated data is given below:

Table1. Simulated data

Number i	**Time interval (T_{i-1}, T_i)**	**Type – I Failures: x_{1i}**	**Type – II Failures: x_{2i}**	**Withdrawals R_i**
1	0-2	5	9	3
2	2-4	11	22	2
3	4-6	12	37	0
4	6-8	5	36	0
5	8-10	12	30	0
6	10-12	2	18	1
7	12-14	5	16	1
8	14-16	1	12	1
9	16-18	2	1	0
10	18-20	2	1	53

As per our notations we have $S_i = T_i^2 - T_{i-1}^2$, m =10, T_m = 20, T_0 = 0, n = 300, r_m= 53.

Solving the equations (3.8) to (3.10) of section 3 we find MLEs of θ_1 and θ_2 as.

$$\widehat{\theta_1} = 18.66779919 \quad and \quad \widehat{\theta_2} = 10.44707906 \qquad ..(8.1)$$

From (3.12) we get MLE for survival function at time t_0 = 6 as

$$\hat{S}(t_0 = 6) = 0.80527175$$

Using (4.1) the asymptotic variance –covariance matrix of the MLE s for parameter θ_1 and θ_2 is given by

$$\begin{pmatrix} V(\widehat{\theta_1}) & Cov(\widehat{\theta_1}, \widehat{\theta_2}) \\ Cov(\widehat{\theta_1}, \widehat{\theta_2}) & V(\widehat{\theta_2}) \end{pmatrix} = \begin{pmatrix} 1.52975785 & 0.00054475 \\ 0.00054475 & 0.11182348 \end{pmatrix}$$

Hence the asymptotic standard errors of the MLEs of the parameters will be

$$SE(\widehat{\theta_1}) = 1.2368338 \qquad \text{and } SE(\widehat{\theta_2}) = 0.3344002$$

Hence from (4.5) we get $SE(\hat{S}(t_0 = 6)) = 0.0123255$.

The 95 % asymptotic confidence interval for θ_1, θ_2 and $S(t_0)$ are obtained from the results of Section 6 respectively as

(16.243604993, 21.0919934), (9.7916547, 11.1025034) and (0.7811137, 0.8294297).

The non parametric point and interval estimates of survival function at time t_0 = 6 are obtained using the results of section 7 as

$\hat{S}(t_0)$ = 0.6757713 , AsyV($\hat{S}(t_0)$) = 0.0006664 and (0.6232762, 0.7241884).

ISSN:0971-1260
Vol-22-Issue-10-November-2019

To apply bootstrap confidence interval estimation for the parameters we have made 1000 simulations based on the MLEs of θ_1 and θ_2 given in (8.1) and keeping the other values as it is m =12, T_m = 20, T_0 = 0, n = 300.

The summary statistics for our simulation are given in the following table:

Table 2. Summary statistics for simulation

	min	0.025	0.25	0.50	0.75	0.975	max
$\widehat{\theta_1}$	19.622169	21.20134	23.61966	24.70123	26.01624	29.46387	32.26648
$\widehat{\theta_2}$	10.768155	11.18613	11.74717	12.12728	12.41849	13.17082	13.83739
$\hat{S}(t_0)$	0.829262	0.836793	0.850758	0.858274	0.865495	0.876099	0.886739
$T\widehat{\theta_1}$	13.577909	30.47288	49.66967	55.69599	61.90557	72.78354	77.29356
$T\widehat{\theta_2}$	17.203020	37.52373	61.16291	75.41570	85.62381	108.0892	124.6738
$T\hat{S}(t_0)$	39.872413	54.39327	85.13249	102.8363	123.7370	156.0913	189.3864

Based on the simulated results the confidence interval based on MLE and bootstrap confidence intervals for parameters and survival function are computed using the methods described in the Section 6, which are given in the following table.

Table 3. Estimates and confidence intervals for the parameters based on MLE and Boot strap

	θ_1	θ_2	$S(t_0 = 6)$
ML estimate	24.875572	12.106256	0.858116
CI by MLE	(21.136642, 28.614503)	(11.354999, 12.857514)	(0.840715, 0.875516)
CI by Percentile Bootstrap	(11.827594, 16.437009)	(8.286605, 9.756848)	(0.740169, 0.774937)
CI by Bootstrap - t	(18.667799, 18.667799)	(10.807864, 11.486343)	(0.886686, 1.038906)

The following tables gives estimate of survival (Reliability) function ($S(t_0)$) of an item at time t_0 = 6. Its asymptotic variance and asymptotic 95% confidence interval of $S(t_0)$ based on MLE and nonparametric estimation based on the results obtained in the sections 6 and 7.

Table 4. Estimate of survival (Reliability) function ($S(t_0)$), Its asymptotic variance and asymptotic confidence interval using MLE

t_0	$\hat{S}(t_0)$	AsyV($\hat{S}(t_0)$)	95% Confidence interval for $\hat{S}(t_0)$		Length of CI
			Lower limit	Upper limit	
2	0.983135	1.000595E-06	0.981178	0.985091	0.003913
4	0.934237	1.5866486E-05	0.926446	0.942028	0.015582
6	0.858116	7.9131704E-05	0.840715	0.875516	0.034801
8	0.761907	0.000244914	0.731293	0.792521	0.061228
10	0.653970	0.000582062	0.606771	0.701168	0.094397
12	0.542692	0.001167934	0.475828	0.609556	0.133728
14	0.435451	0.002081335	0.346182	0.524720	0.178538

ISSN:0971-1260
Vol-22-Issue-10-November-2019

16	0.337889	0.003395180	0.223861	0.451916	0.228055
18	0.253586	0.005169481	0.112866	0.394305	0.281439
20	0.184106	0.007445216	0.015209	0.353004	0.337795

Table 5. Estimate of survival (Reliability) function ($S(t_0)$), Its asymptotic variance and asymptotic confidence interval using nonparametric estimation.

t_0	$\hat{S}(t_0)$	Asy V($\hat{S}(t_0)$)	95% Confidence interval for $\hat{S}(t_0)$		Length of CI
			Lower limit	Upper limit	
2	0.976260	7.700103E-05	0.951408	0.988555	0.037147
4	0.910230	0.000206	0.877939	0.934540	0.056601
6	0.811536	0.000461	0.765828	0.850039	0.084211
8	0.696814	0.000673	0.643705	0.745126	0.101421
10	0.581034	0.000794	0.524990	0.635057	0.110067
12	0.474394	0.000824	0.418689	0.530746	0.112057
14	0.387188	0.000791	0.333711	0.443544	0.109833
16	0.318880	0.000729	0.268432	0.373987	0.105555
18	0.267993	0.000663	0.220614	0.321385	0.100771
20	0.232169	0.000606	0.187478	0.283874	0.096396

Here we see that time up to 10 hours asymptotic variance of the estimate of survival function and length of confidence interval based on MLE are smaller than that of based on non parametric estimation and then after the situation becomes reverse.

Acknowledgements

The authors are grateful to the anonymous referees and the Editor for their time and patience to review the paper.

References

Aggarwala, R.(2001): Progressive interval censoring: Some mathematical results with applications to inference, Communi. in Statistics, Theo. & Meth., Vol. 30(8 & 9), pp.1921-1935.

Bhattacharya, S.K. and Tyagi, R.K(1990): Bayesian survival analysis based on the Rayleigh model, Trabajos de Estadistical, Vol.5, pp. 81-92.

Boardman, T. J. and Kendell, P. J.(1970): Estimation in compound failure models, Technometrics, Vol.12, pp.891-908.

Dayer, D,D and Whisenand, C.W. (1973): Best linear unbiased estimator of the parameter of the Rayleigh distribution, IEEE Trans on Reliab, Vol. R-22, pp. 27-34 & 455-466.

Gadhvi N. K. and Bhimani, G.C.(2016): Inference for Exponential Competing Risk Failure Model, Journal of Advanced Statistics, Vol. 1(2), pp. 76-86.

Gadhvi N.K. and Bhimani, G.C.(2015): Inference Under Progressive Interval Censoring for Discrete Competing Risk Failure Model, Mathematical Theory & Modeling, Vol. 5(2), pp. 153-161.

ISSN:0971-1260
Vol-22-Issue-10-November-2019

Hoel, D. G.(1972): A representation of mortality data by competing risks, Biometrics, Vol 28, pp. 475- 488.

Odell, P. M.; Anderson, K. M. and D’agostino, R. B., (1992), Maximum likelihood estimation for interval censored data using a Weibull based accelerated failure model, Biometrics, Vol. 48, pp.951-959.

Patel, M. N. and Gajjar, A. V.(1992): Maximum likelihood estimation in compound exponential failure model with changing failure rates from Type-I Progressively censored and group censored samples, Communi. in Statistics, Theo. & Meth., Vol .21(1), pp. 2899-2908.

Patel, N.W. and Patel, M.N.(2007): Maximum likelihood estimation in compound geometric failure model with changing parameters from Type-I two-stage Progressively censored and group censored samples, Communi. in Statistics, Theo. & Meth., Vol. 36, pp. 2367-2375**.**

Polovlo, A.M.(1968): Fundamental of Reliability theory, Academic press.

Rao, M. B.(1998): Interval-censored Type-II plan - a knell of the traditional Type-II plan? In IISA International conference 1998, Abstracts, McMaster University, 55.

Rayleigh, L.(1880): On the resultant of a large number of vibrations of some pithead of arbitrary phase philosophical Magazine, Vol.10, pp. 73-80.

Samuelson, S.O. and Kongerud, J.(1994): Interval-censoring longitudinal Data of respiratory of methods, Statistics in Medicine, Vol. 13, pp.1771-1780.

Scallan, A. J. (1999): Regression modelling of interval-censored failure time data using the Weibull distribution, Journal of Applied Statistics, Vol. 26(5), pp. 613-618.

Shah, J.B. and Patel, M.N. (2008)[a] : Bayes estimation under exponentially weighted minimum expected loss function from multiply Type-II censored Rayleigh data, Statistics, Vol.42(5), pp. 429-434.

Shah, J.B. and Patel, M.N. (2008)[b] : Bayes estimation under asymmetric loss function from multiply Type-II censored Rayleigh data, Jr. of Indian Statistical Association, Vol.46(1), pp. 67-78.

Siddiqui, M.M.(1962): Some problem connected with Rayleigh distribution. Jr. Res.Net. Busstand, Vol.66D, pp. 167-174.