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Common Fixed Point Theorems In Fuzzy 3-Metric Space

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Abstract: In this paper, we prove some common fixed point theorems in fuzzy metric space and in fuzzy 3-metric space for weak-compatible mappings and semi-compatible mappings by improving the conditions of sharma[13] and singh[11].

Key words: Fuzzy metric space, fuzzy 3-metric space, t- norm, common fixed points, weak-compatible mappings, semi-compatible mappings.

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1. Introduction: After introduction of fuzzy sets by Zadeh [17], many researchers have defined fuzzy metric space in different ways and used in various field. Sessa [12] introduced a generalization of commutativity so called weak commutativity.Jungck [5] introduced more generalized commutativity which is called compatibility in metric space and proved common fixed point theorems.

In fuzzy metric space Vasuki [15] gives the concept of R weakly commuting maps. Concept of compatible mappings was introduced by Singh and Chauhan [9]. Singh and Jain [10] introduced the concept of a pair of semi compatible self maps in a fuzzy metric space to establish a fixed point theorem for four self mappings, which is extension of Vasuki [15] and also alternate result of Grabiec [3] and Vasuki [14].

Gahler [2] introduced the concept of 2-metric space, whose abstract properties were suggested by the area function in Euclidean space. Wenzi [16] initiated the concept of probabilistic 2-metric spaces. Recently Sharma [13], Singh and Jain [11] proved certain common fixed point theorems in fuzzy metric spaces, fuzzy 2-metric spaces and fuzzy 3-metric spaces.

In this paper we prove three fixed point theorems for three self maps which generalize the results of [11, 13] in fuzzy metric spaces, and fuzzy 3-metric spaces respectively by using the concept of semi-compatibility and weak-compatibility.

2. Preliminaries:

Definition 2.1: A binary operation $*$: $[0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if $([0, 1], *)$ is an abelian topological monoid with unit 1 such that $a*b = c*d$ whenever $a \leq c$ and $b \leq d$ for all a, b, c and $d \in [0, 1]$. Examples of t-norm are $a * b = ab$ and $a * b = \min \{a, b\}$.

Definition 2.2(Kramosil and Michalek [6])**:** A triplet $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm and M is a fuzzy set on $X \times X \times [0,\infty) \to [0, 1]$ satisfying the following conditions:

For all $x, y, z \in X$ and $s, t > 0$:

(FM - 1) $M (x, y, 0) = 0$;

(FM - 2) $M (x, y, t) = 1$, for all $t > 0$ iff $x = y$;

(FM - 3) $M (x, y, t) = M (y, x, t)$;

(FM - 4) $M (x, y, t) * M (y, z, s) = M (x, z, t + s)$;

(FM - 5) $M (x, y,.): [0, \infty) \to [0, 1]$ is left continuous.

In this paper (X, M, *) is considered to be the fuzzy metric space with condition (FM - 6) $\lim_{t \to \infty} M (x, y, t) = 1$.

Note that M (x, y, t) can be thought of as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M (x, y, t) = 1$ for all $t > 0$. The following example shows that every metric space induces a fuzzy metric space.

Example2. 3: Let (X, d) be a metric space. Define $a*b = \min \{a, b\}$

and for all $x, y \in X$, $M (x, y, t) = t/ (t + d(x, y))$ for all $t > 0$ and $M (x, y, 0) = 0$. Then (X, M, *) is a fuzzy metric space. It is called the fuzzy metric space induced by the metric d.

Lemma 2.4 (Grabiec [3])**:** For all $x, y \in X$, M (x, y,.) is a non - decreasing function.

Definition2.5 (Grabiec [3])**:** Let $(X, M, *)$ be a fuzzy metric space. A sequence {xn} in X is said

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to convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

The sequence $\{x_n\}$ is said to be a Cauchy sequence in X, if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for all $t > 0$ and $p > 0$.

The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Remark 2.6: Since $*$ is continuous, it follows from (FM - 4) that the limit of a sequence in a fuzzy metric space is unique, if it exists.

Definition 2.7: A function M is continuous in fuzzy metric space iff whenever $\{x_n\} \to x$ and $\{y_n\} \to y$ then $\lim_{n\to\infty} M(x_n, y_n, t) = M(x, y, t)$ for each $t > 0$.

Definition 2.8: A binary operation $*: [0, 1] \times [0, 1] \times [0, 1] \to [0, 1]$ is called a continuous t-norm if ([0, 1], *) is an abelian topological monoid with unit 1 such that $a * b * c = d * e * f$ whenever $a \leq d$, $b \leq e$ and $c \leq f$, for all a, b, c, d, e and

Definition 2.13 9(Sharma [13])**:** The 3-tuple (X, M, $*$) is called a fuzzy 3-metric space if X is an arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set in

$X^4 \times [0, \infty)$ satisfying the following conditions, for all $x, y, z, u, w \in X$ and t, t1, t2, t3, t4> 0:

(FM''- 1) $M(x, y, z, w, 0) = 0$;

(FM''- 2) $M(x, y, z, w, t) = 1$, for all $t > 0$, iff atleast two of the four points are equal

(FM''- 3) $M(x, y, z, w, t) = M(y, x, z, w, t) = M(w, z, x, y, t) = \ldots$(symmetry)

(FM''-4) $M(x, y, z, t1+t2+t3+t4) = M(x, y, z, u, t1) * M(x, y, u, w, t2) * M(x, u, z, w, t3) * M(u, y, z, w, t4)$;

(FM''- 5) $M(x, y, z, w, .): [0, \infty) \to [0, 1]$ is left continuous.

Definition 2.10: Let (X, M, $*$) be a fuzzy 3-metric space. A sequence $\{x_n\}$ in X is said to convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, a, b, t) = 1$ for all $a, b \in X$ and for all $t > 0$.

The sequence $\{x_n\}$ is said to be a Cauchy sequence in X, if $\lim_{n\to\infty} M(x_n, x_{n+p}, a, b, t) = 1$ for all $a, b \in X$, $t > 0$ and $p > 0$.

The space is said to be complete if every Cauchy sequence in it converges to a point of it.

Definition 2.11: A function M is continuous in fuzzy 3-metric space iff whenever $\{x_n\} \to x$ and $\{y_n\} \to y$ then $\lim_{n\to\infty} M(x_n, y_n, a, b, t) = M(x, y, a, b, t)$ for all $a, b \in X$ and for each $t > 0$.

Lemma2.12: (Mishra [7])**:** Let (X, M,*) be a fuzzy metric space. If there exists a number $k \in (0, 1)$ such that for all $x, y \in X$ and $t > 0$, $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

Lemma 2.4, 2.6 and remark 2.12 hold fuzzy 3-metric also.

3. Compatible Mappings in Fuzzy Metric Space:

Definition 3.1 (Mishra [7])**:** Let A and B be mappings from a fuzzy metric space, (X, M, $*$) into itself. The mappings are said to be compatible if

$\lim_{n\to\infty} M(AB x_n, BA x_n, t) = 1$ for all $t > 0$, whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

Definition 3.2: Let A and B be mappings from a fuzzy metric space (X, M, *) into itself. The mappings are said to b e weak-compatible if they commute at their coincidence points i.e. $Ax = Bx$ implies $ABx = BAx$.

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Definition 3.3 ([12]): Let A and B be mappings from a fuzzy metric space, (X, M, *) into itself. The mappings are said to be semi- compatible if $\lim_{n\to\infty} M(ABx_n, Bx, t) = 1$ for all $t > 0$, When- ever, $\{x_n\}$ is a sequence in X such that, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

4. Compatible Mappings in Fuzzy 3 Metric Space:

Definition 4.1: Let A and B be mappings from a fuzzy 3-metric space, (X, M, $*$) into itself. The mappings are said to be compatible if

$M(AB\, x_n, BA\, x_n, a, b, t) = 1$ for all $t > 0$ and for all $a, b \in X$. Whenever, $\{x_n\}$ is a sequence in X such that

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$

Definition 4.2: Let A and B be mappings from a fuzzy 3-metric space, (X, M, $*$) into itself. The mappings are said to b e semi-compatible if

$\lim_{n\to\infty} M(ABx_n, Bx, a, b, t) = 1$ for all $t > 0$ and for all $a,b \in X$. Whenever, $\{x_n\}$ is a sequence in X such that,

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = x \in X$.

Remark 4.3: Let (A, B) be a pair of self mappings of a fuzzy metric space (X, M, $*$). Then (A, B) is commuting implies (A, B) is compatible. Also (A, B) is compatible implies (A, B) is weak-compatible but the converse is not true.

Remark 4.4 [10]**:** A pair of self maps is given which is commuting hence compatible, weak-compatible but it is not semi-compatible.

Remark 4.5 [10]**:** Let A and B be self maps on a fuzzy metric space (X, M, *). If B is continuous then (A, B) is semi-compatible if (A, B) is compatible.

It is easy to verify that the above remark is true for fuzzy 3-metric spaces also.

5. Main Results

Theorem 5.1: Let A, S and T is self mappings of a complete FM- space (X, M, $*$) satisfying:

(1) $A(X) \subset S(X)$ n $T(X)$,

(2) A or S is continious,

(3) Pair {A, S} is semi compatible and {A, T} is weak compatible,

(4) there exists $k \in (0, 1)$ such that for all $x,y \in X$ and $t > 0$

$$M(Ax,Ay,t) \geq \operatorname{Min}\{M(Sx,Ty,t),M(Sx,Ax,t),M(Sx,Ay,t),M(Ty,Ay,t)\},$$

(5) $\lim_{n\to\infty} M(x, y, t) = 1$, for all $x,y \in X$ and $t > 0$.

Then A, S and T have unique common fixed point in X.

Proof. Let $x \in X$ be any arbitrary point. Since $A(X) \subset S(X)$ then there exists a point $x1 \in X$ such that $Ax_0 = Sx_1$. Also since $A(X) \subset T(X)$, there exists another point $x_2 \in X$ such that $Ax_1 = Tx_2$. Inductively construct sequences $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n+1}= Sx_2\, n+1= Ax_{2n}$ and $y_{2n+2}= Tx_{2n+2}= Ax_{2n+1}$, for n=0,1,2,…

Now put $x=x_{2n+1}$ and $y=x_{2n}$ in (4), we get

$$M(Ax_{2n+1},Ax_{2n},kt) \geq \operatorname{Min}\{M(Sx_{2n+1},Tx_{2n},t),M(Sx_{2n+1},Ax_{2n+1},t),M(Sx_{2n+1},Ax_{2n},t), M(Tx_{2n},Ax_{2n},t)\}$$

$$\geq \operatorname{Min}\{M(y_{2n+1}, y_{2n}, t), M(y_{2n+1}, y_{2n+2}, t), M(y_{2n+1}, y_{2n+1}, t), M(y_{2n}, y_{2n+1}, t)\}$$

$M(y_{2n+2}, y_{2n+1}, kt) \geq M(y_{2n+1}, y_{2n}, t)$

Similarly,

$M(y_{2n+3}, y_{2n+2}, kt) \geq M(y_{2n+2}, y_{2n+1}, t)$

Thus we have

$M(y_{n+1},y_{n+2},kt) \geq M(y_n, y_{n+1}, t)$ for n=1,2,3,…

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Hence $M(y_{n+1},y_{n+2},kt) \geq M(y_n, y_{n+1}, t/k)$ for n=1,2,3,…

Now $M(y_n,y_{n+1},t) \geq M(y_{n-1},y_n,t/k) \geq M(y_{n-2},y_{n-1},t/k^2) \geq \ldots \geq M(y_0,y_1,t/k^m) \rightarrow 1$ as $m \rightarrow \infty$

Hence $\lim_{n\to\infty} M(y_n,y_{n+1},t) = 1$, for all t >0.

Now for any positive integer p

$M(y_n,y_{n+p},t) \geq M(y_n,y_{n+1},t/p) * M(y_{n+1},y_{n+2},t/p) * \ldots * M(y_{n+p-1}, y_{n+p}, t/p)$

Therefore $\lim n \rightarrow \infty$ $M(y_n,y_{n+p},t) \geq 1 * 1 * \ldots * 1 = 1$

Hence by definition (4) $\{y_n\}$ is a Cauchy sequence and by the completeness of X, $\{y_n\}$ converges to a point $z \in X$ and so its subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n}\}$ also converges to z.

Case I: A is continuous

In this case $AAx_{2n} \rightarrow Az$, $ASx_{2n} \rightarrow Az$

and semi-compatibility of the pair (A, S) gives, $\lim n \rightarrow \infty$ $ASx = Sz$.

As limit of a sequence in fuzzy metric space is unique, we have

$$Az = Sz$$

Step I: Putting $x = z$, $y = x_{2n}$ in (4), we get

$M(Az,Ax_{2n+1},kt) \geq \text{Min}\{M(Sz,Tx_{2n},t),M(Sz,Az,t),M(Sz,Ax_{2n+1}, t), M(Tx_{2n}, Ax_{2n+1}, t)\}$

Taking $\lim n\rightarrow\infty$, we get

$M(Az, z, kt) \geq \text{Min}\{M(Sz, z, t), M(Sz, Az, t), M(Sz, z, t), M(z, z, t)\} = M(Az, z, t)$, for all $t > 0$.

By lemma 2.12, Az = z. Thus Az = S z = z.

Step 2: As $A(X) \subset T(X)$, there exists $u \in X$ such that z = Az = Tu.

Putting $x = x_{2n+1}$, $y = u$ in (4), we get

$M(Ax_{2n+1}, Au, kt) \geq \text{Min}\{M(Sx_{2n+1},Tu, t),M(Sx_{2n+1}, Ax_{2n}, t),$

$M(Sx_{2n+1},Au, t),M(Tu,Au,t)\}$

Taking $\lim n\rightarrow\infty$, we get

$M(z, Au, kt) \geq \text{Min}\{M(z,Tu, t), M(z, z, t), M(z, Au, t), M(Tu, Au, t)\}$

$= M(Au, z, t)$, for all $t > 0$.

By lemma2.12, we get, z = Bu = Tu and the weak compatibility of (A,T) gives

TAu =ATu i.e. Tz = Az .

Az = Sz = Tz = z

Hence z is a common fixed point of A, S and T.

Case II: S is continuous

In this case $SAx_{2n} \rightarrow Sz$, $SSx_{2n} \rightarrow Sz$

And semi-compatibility of the pair (A, S) gives $\lim n \rightarrow \infty$ $ASx_{2n} = Sz$

Step3: Putting $x = Sx_{2n+1}$, $y = x_{2n}$ in (4), we get

$M(ASx_{2n+1},Ax_{2n}, kt) \geq \text{Min}\{M(SSx_{2n+1},Tx_{2n}, t),M(SSx_{2n+1}, ASx_{2n+1}, t),$

$M(SSx_{2n+1}, Ax_{2n}, t),M(Tx_{2n}, Ax_{2n}, t)\}$,

Taking $\lim n\rightarrow \infty$, we get

$M(Sz, z, kt) \geq \text{Min}\{M(z, z, t),M(z, Sz, t),M(Sz, z, t), M(z, z, t)\}$

$\geq M(Sz, z, t)$, for all $t > 0$.

By lemma 2.12, we get, Sz = z

Step 4: Putting $x= z$, $y = x_{2n}$ in (4), we get

$M(Az,Ax_{2n},kt) = \text{Min}\{M(Sz,Tx_{2n}, t),M(Sz, Az, t),M(Sz ,Ax_{2n}, t), M(Tx_{2n}, Ax_{2n}, t)\}$,

Taking $\lim n\rightarrow\infty$, we get

$M(Az, z, kt) = \text{Min}\{M(z, z, t), M(z, z, t), M(z, z, t), M(z, z, t)\}$

$=M(Az, z, t)$ for all $t > 0$.

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By lemma 2.12, we get, Az = z.. Therefore, Az = Sz = z . Now, apply step 2 and 3 of case 1 to get Tz = Az = z.

Thus, z = Az = Sz = Tz .

I.e. z is a common fixed point of A, S and T in this case also.

Uniqueness: Let u be another common fixed point of A, S and T.

Then u = Au = S u = T u.

Putting x = z and y = u in (4) we get,

M(Az,Au,kt) = Min{ M(Sz,Tu,t),M(Sz,Au,t),M(Sz,Au,t), M(Tz,Au,t)}

i.e. M (z,u,kt) = M (z,u,t) Which yields z = u and therefore z is the unique common fixed point of the three self maps A, S and T.

Example 1: Let (X, d) be the metric space with X = [0, 2] and

d(x, y) = |x - y|. Let (X, M, *) be the induced fuzzy metric space, with

a $*$ b = Min{a, b} and M(x, y, t) = t / t+ |x - y|

A, S and T as follows

A(X) =2,

Sx = {0 if x $\in$ [0, 2), 2 if x $\in$ [2, ∞),

Tx= {1 if x $\in$ [0, 2), 2 if x $\in$ [2, ∞).

Now we see that A(X)={2} , S(X)={0,2} ,T(X)={1,2} and so

A(X) $\subset$ S(X) n T(X).

Also If xn = 2-1/2n then, we have

Lim n$\to\infty$ M(ASx_n, Sx, t) = M(2, 2, t) = 1

And Lim n$\to\infty$ M(ATx_n,TAx_n,t) = M(2,2, t) = 1.

Hence, the pairs of mappings (A,S) is semi compatible and (A,T) is weakly compatible.

Case 1. If x, y $\in$ [0, 2), then we have M(Ax, Ay, t)=1, M(Sx,Ty, t)= t/ t+1

M(Sx, Ax, t)=t/ t+2, M(Sx, Ay, t)=t/ t+2, M(Ty, Ay, t)= t/ t+1

and so

M(Ax,Ay,t) $\geq$ Min{M(Sx,Ty, t),M(Sx, Ax, t),M(Sx, Ay, t),M(Ty, Ay, t)}.

Case 2. If x, y $\in$ [2, ∞), then we have

M(Ax, Ay, t)=1, M(Sx, Ty, t)=1,

M(Sx, Ax, t)=1, M(Sx, Ay, t)=1, M(Ty, Ay, t) = 1

and so

M(Ax, Ay, t)=Min{M(Sx,Ty, t),M(Sx, Ax, t),M(Sx, Ay, t),M(Ty, Ay, t)}

Case 3. If x $\in$ [0, 2), y $\in$ [2,∞) then we have

M(Ax, Ay, t)=1, M(Sx, Ty, t)= t/ t+2

M(Sx, Ax, t)= t/ t+2, M(Sx, Ay, t)= t/ t+2, M(Ty, Ay, t)=1

and so

M(Ax, Ay, t) $\geq$ Min{M(Sx, Ty, t),M(Sx, Ax, t),M(Sx, Ay, t),M(Ty, Ay, t)}.

Case 4. It is same as in case 3 as M is a symmetric function.

Hence in all cases, we see that all the conditions of Theorem 6.1 are satisfied. And 2 is the common fixed point of A, S and T.

Corollary 5.2: Let A, S and T is self mappings of a complete FM- space

(X, M, $*$) satisfying:

(1) A(X) $\subset$ S(X) n T(X),

(2) A or S is continious,

(4) there exists k $\in$ (0, 1) such that for all x,y $\in$ X and t > 0

M(Ax, Ay, kt) $\geq$ Min{M(Sx, Ty, t),M(Sx, Ax, t),M(Sx, Ay, t),M(Ty, Ay, t)},

(5) Lim n$\to\infty$ M(x, y, t) = 1, for all x,y $\in$ X and t > 0.

(6) Pair {A, S} is compatible and {A, T} is weak compatible.

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Then A, S and T have unique common fixed point in X.

Proof: The proof follows from remark (5.5) and theorem (5.1).

If we take S = T = I , the identity map on X in theorem (5.1), then the conditions (1), (2),(3) and (5) are trivially satisfied., we get the following result :

Corollary 5.3: Let A be a self maps of a complete FM-space (X, M, *) satisfying:

(7) there exists $k \in (0, 1)$ such that for all $x,y \in X$ and $t > 0$

$$M(Ax, Ay, kt) \geq Min\{M(x, y, t), M(x, Ax, t), M(x, Ay, t), M(y, Ay, t)\},$$

Then A has a unique common fixed point in X.

In the following we extend theorem 5.1 to fuzzy 3-metric space as follows:

Theorem 5.4: Let A, S and T be self mappings of a complete fuzzy 3-metric space (X, M, $*$) satisfying (1), (2), (3) and

(8) there exists $k \in (0, 1)$ such that for all $x,y,a,b \in X$ and $t > 0$

$$M(Ax,Ay,a,b,t) \geq Min\{M(Sx,Ty,a,b,t), M(Sx,Ax,a,b,t), M(Sx,Ay,a,b,t), M(Ty,Ay,a,b,t)\},$$

(9) $\lim_{n\to\infty} M(x, y, a,b,t) = 1$ for all $x,y,a,b \in X$ and $t > 0$.

Then A, S and T have unique common fixed point in X.

Proof: Let $x \in X$ be any arbitrary point. Since $A(X) \subset S(X)$ then there exists a point $x_1 \in X$ such that $Ax_0 = Sx_1$. Also since $A(X) \subset T(X)$, there exists another point $x2 \in X$ such that $Ax_1 = Tx_2$. Inductively construct sequences {yn} and {xn} in X such that $y_{2n+1} = Sx_{2n+1} = Ax_{2n}$ and $y_{2n+2} = Tx_{2n+2} = Ax_{2n+1}$, for n=0,1,2,…

Now put $x=x_{2n+1}$ and $y=x_{2n}$ in (8), we get

$$M(Ax_{2n+1},Ax_{2n},a,b,kt) \geq Min\{M(Sx_{2n+1},Tx_{2n},a,b,t), M(Sx_{2n+1},Ax_{2n+1},a,.b.t), M(Sx_{2n+1},Ax_{2n},t), M(Tx_{2n},Ax_{2n},t)\}$$

$$\geq Min\{M(y_{2n+1},y_{2n},a,b,t), M(y_{2n+1}, y_{2n+2}, a,b.t), M(y_{2n+1}, y_{2n+1}, a,b.t), M(y_{2n}, y_{2n+1}, a,b.t)\}$$

$$M(y_{2n+2},y_{2n+1},a,b,kt) \geq M(y_{2n+1},y_{2n}, a,b,t)$$

Similarly,

$$M(y_{2n+3}, y_{2n+2}, a,b,kt) \geq M(y_{2n+2}, y_{2n+1}, a,b,t)$$

Thus we have

$$M(y_{n+1},y_{n+2}, a,b,kt) \geq M(y_n, y_{n+1}, a,b,t)$$

for n=1,2,3,…

Hence $M(y_{n+1},y_{n+2},a,b, t) \geq M(y_n, y_{n+1}, a,b,t/k)$ for n=1,2,3,…

Now

$$M(y_n,y_{n+1},a,b,t) \geq M(y_{n-1},y_n,a,b,t/k) \geq M(y_{n-2},y_{n-1},a,b,t/k^2) \geq \ldots \geq M(y_0,y_1,a,b,t/k^m) \to 1$$

as $m \to \infty$

Hence $\lim_{n\to\infty} M(y_n,y_{n+1},a,b,t) = 1$, for all $t > 0$.

For any positive integer p, we prove that $\lim_{n\to\infty} M(y_n,y_{n+p},a,b,t)=1$, for all $t > 0$

We prove it by induction on p.

Clearly it is true for p = 1.

Suppose above is true for p = m.

i.e. $\lim_{n\to\infty} M(y_n,y_{n+m},a,b.t)=1$ for all $t > 0$.

Now using (FM'4)

$$M(y_n, y_{n+m+1}, a,b,t) = M(y_n, y_{n+m}, a, b,t/3) * M(y_n, y_{n+m}, y_{n+m+1}, t/3) * M(y_{n+m}, y_{n+m+1},a, b,t/3)$$

Therefore,

Lim $n\to\infty$ $M(y_n, y_{n+m+1}, a,b, t) = 1 * 1 * 1 = 1$.

Hence above is true for p = m + 1. Thus, it holds for all p and we get

$\{y_n\}$ is a Cauchy sequence in X which is complete. Therefore, $\{y_n\}$ converges to $z\in X$ and and so its subsequences $\{Ax_{2n}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n}\}$ also converges to z.

Case I: A **is continuous**

In this case $AAx_{2n}\to Az$, $ASx_{2n}\to Az$

And semi-compatibility of the pair (A, S) gives,

Lim $n\to\infty$ $ASx= Sz$.

As limit of a sequence in fuzzy metric space is unique, we have

$$Az = S z$$

Step I: Putting $x = z$, $y = x_{2n}$ in (8), we get

$M(Az,Ax_{2n+1},a,b,kt) \geq \text{Min}\{M(Sz,Tx_{2n},a,b,t),M(Sz,Az,a,b,t),M(Sz,Ax_{2n+1},a,b,t),$

$M(Tx_{2n},Ax_{2n+1},a,b,t)\}$

Taking Lim $n\to\infty$, we get,

$M(Az, z, a,b ,k t) \geq \text{Min}\{M(Sz,z,a,b,t), M(Sz,Az,a,b,t), M(Sz, z,a,b,t), M(z,z,a,b,t)\}$

$\geq M(Az,z,a,b,t)$, for all $t > 0$.

By lemma 2.12, Az = z.

Thus Az = S z = z.

Step 2: As $A(X) \subset T(X)$, there exists $u \in X$ such that $z = Az = Tu$.

Putting $x = x_{2n+1}$, $y = u$ in (8), we get

$M(Ax_{2n+1},Au,a,b,kt) \geq \text{Min}\{(Sx_{2n+1},Tu,a,b,t),M(Sx_{2n+1},Ax_{2n},a,b,t),M(Sx_{2n+1},Au,a,b,t),$

$M(Tu,Au,a,b,t)\}$

Taking Lim $n\to\infty$, we get

$M(z, Au,a,b, kt) \geq \text{Min}\{M(z,Tu,a,b,t), M(z,z,a,b,t), M(z,Au,a,,b,t), M(Tu,Au,a,b,t)\}$

$= M(Au,z,a,b, t)$, for all $t > 0$.

By lemma 2.12, we get, z = B u = T u and the weak compatibility of (A,T) gives

$$TAu =ATu \text{ i.e. } Tz = Az .$$

$$Az = Sz =Tz=z$$

Hence z is a common fixed point of A, S and T.

Case II: S is continuous

In this case $SAx_{2n}\to Sz$, $SSx_{2n}\to Sz$

And semi-compatibility of the pair (A, S) gives

Lim $n \to \infty$ $ASx_{2n} = Sz$

Step3: Putting $x = Sx_{2n+1}$, $y = x_{2n}$ in (3.14), we get

$M(A Sx_{2n+1},Ax_{2n},a,b,kt) \geq \text{Min}\{M(SSx_{2n+1},Tx_{2n},a,b, t),M(S Sx_{2n+1},ASx_{2n+1},a,b,t),$

$M(SSx_{2n+1},Ax_{2n},a,b,t),$

$M(Tx_{2n},Ax_{2n},a,b,t)\}$,

Taking Lim $n\to\infty$, we get

$M(Sz,z,a,b,k t) \geq \text{Min}\{M(z,z,a,b,t), M(z,Sz,a,b,t), M(Sz,z,a,b,t), M(z,z,a,b,t)\}$

$= M(Sz,z,a,b,t)$, for all $t > 0$.

By lemma 2.12, we get, Sz = z

.**Step 4:** Putting $x= z$, $y = x_{2n}$ in (4), we get

$M(Az,Ax_{2n},a,b,kt) \geq \text{Min}\{M(Sz,Tx_{2n},a,b,t),M(Sz, Az, a,b,t),M(Sz ,Ax_{2n},a,b,t),$

$M(Tx_{2n}, Ax_{2n},,a,b,t)\}$

Taking Lim $n\to\infty$, we get

$M(Az,z,a,b,kt) \geq \text{Min}\{M(z,z,a,b,t), M(z,z,a,b,t), M(z,z,a,b,t), M(z,z,a,b,t)\}$

$= M(Az,z,a,b,t)$ for all $t > 0$.

By lemma 2.12, we get, Az = z. Therefore, Az = S z = z. Now, apply step 2 and 3 of case 1 to get Tz = Az = z.

Thus, z = Az = Sz = Tz.

i.e. z is common fixed point of A, S and T in this case also.

Uniqueness: Let u be another common fixed point of A, S and T.

Then u = Au = S u = T u.

Putting x = z and y = u in (4) we get,

M(Az,Au,a,b,kt) = Min{ M(Sz,Tu,a,b,t),M(Sz,Au,a,b,t),M(Sz,Au,a,b,t), M(Tz,Au,a,b,t)}

i.e. M (z,u,a,b,kt) = M (z,u,a,b,t) Which yields z = u and therefore z is the unique common fixed point of the three self maps A, S and T.

Theorem 5.5: Let A, S and T be self mappings of a complete fuzzy 3-metric space (X, M, *) satisfying (1),(2),(8),(9)and (10) Pair {A, S} is compatible and {A, T} is weak compatible,

Then A, S and T have unique common fixed point in X.

Proof: The proof follows from remark (5.5)and theorem (5.4).

If we take S = T = I , the identity map on X in theorem 5.4, then the conditions (1), (2),(3) and (9) are trivially satisfied., we get the following result :

Corollary 5.6: Let A be a self maps of a complete FM- space (X, M, $*$) satisfying:

(11) there exists $k \in (0, 1)$ such that for all x,y,a,b $\in X$ and $t > 0$

M(Ax,Ay,a,b,kt) $\geq$ Min{M(x,y,a,b,t),M(x,Ax,a,b,t),M(x,Ay,a,b,t),M(y,Ay,a,b,t)}.

Then A has a unique common fixed point in X.

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