



GROUPS WHOSE INTERSECTION GRAPHS ARE COMPLETE

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ABSTRACT:

In a graph theory we have use basis of the group like abelian group and subgroup for intersection and complete. We characterize certain classes of subgroup intersection graphs are complete. We check all its solvable groups whose intersection graphs are complete. Surrounded the other results, we analysis all abelian groups whose intersection graphs are complete. Finally, we study the intersection graphs of cyclic groups. In this section, we study the groups whose intersection graphs are complete.

KEY WORDS:

Intersection graph, Complete graph, Finite graph, Prime number, Parski groups, cyclic group, isomorphic, torsion-free group, Monomorphism, torsion group

1.PRELIMINARIES

Definition 1.1.



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Let G be a group. The intersection graph of G , $\Gamma(G)$ is a graph whose vertices are all non trivial proper subgroups of G and two distinct vertices H and K are adjacent if and only if $H \cap K \neq 1$.

Suppose that H and K are two subgroups of G . We say that G is the semi-direct product of H and K , written $G = HK$ if $G = HK$, $H \cap K = 1$, and H is normal in G .

The center of G is denoted by $Z(G)$.

Definition 1.2.

An abelian group G is said to be divisible,

If $nG = \{ng : g \in G\} = G$ for every positive integer n .

Notation 1.3.

For a fixed prime p , the quasi-cyclic p -group is denoted by $Z(p^\alpha)$.

For positive integers $m \geq 2$ and $n \geq 3$,

Z_m and Q_{2^n} denoted the group of integers modulo m and the generalized quaternion group of order 2^n respectively.

Let Γ be a graph. The complement of Γ is denoted by $\bar{\Gamma}$.

The set of vertices and the set of edges of Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$ respectively.



Definition 1.4.

A subgroup ω of Γ is called a spanning subgraph if $V(\omega) = V(\Gamma)$.

We denote by $\deg(v)$, the degree of vertex v in Γ .

Definition 1.5.

The girth of Γ , denoted by $gr(\Gamma)$, is the length of its shortest cycle

$(gr \Gamma) = \infty$ (if Γ has no cycles).

Definition 1.6.

An independent set $S \subseteq V(\Gamma)$ is a set whose no two vertices are adjacent. The independent number of Γ is denoted by $\alpha(\Gamma)$ is the cardinality of the largest independent set.

Definition 1.7.

A Γ -free graph is a graph which does not contain Γ as a subgraph.

Definition 1.8.

A null graph is a graph with no edges and a star graph is a graph with a vertex adjacent to all other vertices and there is no extra edge.

A star with n edges is denoted by $K_{1,n}$.



Definition 1.9.

We denoted the complete graph and the path is denoted by $\Gamma \cup \Omega$.

The join of Γ and Ω , $\Gamma \vee \Omega$ is the graph obtained from the disjoint union $\Gamma \cup \Omega$ by adding the edges $\{xy : x \in V(\Gamma), y \in V(\Omega)\}$.

Notation 1.10.

In this paper, We show that $\text{gr}(\Gamma(G)) \in \{3, \infty\}$.

We characterize all solvable groups whose intersection graphs are triangle-free.

Moreover we so that if G is finite and $\Gamma(G)$ is triangle-free, then G is solvable.

Also, we prove that if $\Gamma(G)$ is a triangle-free graph, then it is a disjoint union of some stars.

Among other results, we classify all abelian group whose intersection graphs are complete.

The intersection graph of a finite cyclic group will be studied in the last section.

We show that if $n = (p_1)^{n_1} \dots (p_k)^{n_k}$ and $m = (q_1)^{m_1} \dots (q_t)^{m_t}$, where p_i and q_j are distinct

Primes and n_i and m_j are positive integers.

Then $\Gamma(Z_n) \cong \Gamma(Z_m)$ if and only if $k = t$ and there is a permutation π such that $n_i = m_{\pi(i)}$ for

$i = 1 \dots k$.



Definition 1.11.

A non empty set G together with a binary operation $*$: $G \times G \rightarrow G$ is called a group if the following conditions are satisfied

- 1) $*$ is closure for all $a, b \in G$, $ab \in G$.
- 2) $*$ is associative ie, $a*(b*c) = (a*b)*c$ for all $a, b, c \in G$.
- 3) There exists an element $e \in G$ such that $a*e = e*a = a$ for all $a \in G$,
 e is called the identity element of G .
- 4) For any element a in G there exists an element $a' \in G$ such that
 $a*a' = a'*a = e$, a' is called the inverse of a .

Example :

Z , Q , R and C are groups under usual addition.

Definition 1.12.

Given a group G under a binary operation $*$ a subset H of G is called a subgroup of G if H is also forms a group under the operation $*$.

Example :



The even integers are a subgroup of the additive group of integers.

Definition 1. 13.

A proper subgroup of a group G is a subgroup H which is a Proper subset of G (i.e., $H \neq G$). This is usually represented notationally by $H < G$ read as H is a proper subgroup of G .

Definition 1. 14.

An abelian group is a set G combined with a binary operation $*$: $G \times G \rightarrow G$ ie, $*$ takes the elements of G and returns an element of G satisfies the following properties.

Group axioms :

- 1) Associativity : For any $x, y \in G$ the relation

$$(x*y)*z = x*(y*z).$$

- 2) Identity there exists an element $e \in G$ such that $e*x = x*e = x$ for any $x \in G$, e is called identity element of G .

- 3) Inverse : For any $x \in G$ there exists $y \in G$ such that $x*y = e = y*x$, y is called an inverse of x .

- 4) Closure : For any $x, y \in G$ the relation $x*y \in G$.

- 5) Commutative : For any $x, y \in G$ the relation $x*y = y*x$.



A group G is said to be abelian if $ab = ba$ for all $a, b \in G$.

Example :

$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ usual addition are abelian groups.

Definition 1. 15.

A finite abelian group is a group satisfying the following equivalent.

- 1) It is both finite and abelian.
- 2) It is isomorphic to a direct product of finitely many finite cyclic group.
- 3) It is isomorphic to a direct product of abelian group of prime power order.
- 4) It is isomorphic to a direct product of cyclic groups of prime power order.

Example :

$\mathbb{Z}_{15}$ can be expressed as the direct sum of two cyclic subgroups of two cyclic subgroups of order 3 and 5.

$$\mathbb{Z}_{15} \cong \{0, 5, 10\} \oplus \{0, 3, 6, 9, 12\}.$$

The same can be said for any abelian group of order 15, leading to the remarkable conclusion that all abelian groups of order 15 are isomorphic.

Definition 1. 16.



Given any group G , the group consisting of only the identity element is a subgroup of G and being the trivial group is called the trivial subgroup of G .

The term when referred to

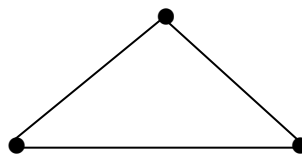
" G has no non trivial proper subgroup" refers to the only subgroup of G being the trivial group $\{e\}$ and the group G itself.

Definition 1. 17.

A graph in which any two distinct points are adjacent is called a complete graph.

Example :

K_3 is called a triangle.



K_3

The complete graph with p points is denoted by K_p

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Theorem 2.1.

Let G be a finite graph. Then $\Gamma(G)$ is a complete graph if and only if

G is isomorphic to Z_p^n , for some prime number p and some integer $n \geq 2$ or Q_2^n for some

Integer $n \geq 3$.

Proof .

If $\Gamma(G)$ is complete, then it is clear G is a p group for some prime number p .

Also, G contains a unique subgroup of order p .

So, by cyclic or generalized quaternion, the other side is obvious.

There are rather exotic infinite non-abelian groups whose intersection graph are complete graph.

For intersection see extended Parski groups.

We know that, For every finite cyclic group of order n , for each divisor d of n there exists a unique subgroup of order d .

Example 2.2.

Consider the cyclic group of order 32 (ie, Z_{32}).



Soln.

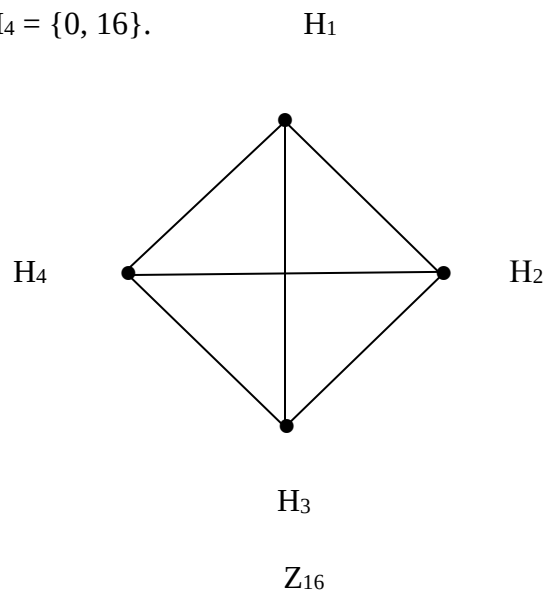
Z_{32} has 4 proper subgroups as follows:

$$H_1 = \{0, 2, 4 \dots 30\},$$

$$H_2 = \{0, 4, 8 \dots 32\},$$

$$H_3 = \{0, 8, 16, 24\},$$

$$H_4 = \{0, 16\}.$$



Theorem 2.3.

Let G be an abelian group. Then $\Gamma(G)$ is a complete graph, if and only if G is isomorphic to a subgroup of Q or $Z(p^a)$, for some prime number p .

Proof.



Assume that $\Gamma(G)$ is a complete graph.

If G is finite, then by theorem, G is a cyclic p group, for some prime number p .

Now, suppose that G is infinite.

First suppose that G is a torsion-free group.

Let $0 \neq x \in G$.

Since $\Gamma(G)$ is a complete graph,

So $\langle x \rangle \cap \langle y \rangle \neq 0$, for each $0 \neq y \in G$.

Thus there two non zero integers m and n such that $mx = ny$.

So we have the monomorphism $\phi : G \rightarrow Q$ with $\phi(0) = 0$ and $\phi(y) = \frac{m}{n}$ for $0 \neq y \in G$.

Therefore G is isomorphism to a subgroup of Q .

Suppose there is an element $0 \neq x \in G$ such that x has a finite order.

Again, since $\Gamma(G)$ is a complete graph $\langle x \rangle \cap \langle y \rangle \neq 0$, for each $0 \neq y \in G$.

Hence G is a torsion group.

Clearly, G is a p -group.

If H is a finite subgroup of G , Then $H = Z_p^n$ or $\Gamma(G)$ is complete and by theorem, we have



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$H = Z_p^n$ for some $n \geq 1$.

Let H and K be two subgroups of G and $|H|$ and $|K|$.

Then HK is cyclic and $H = K$.

Now, G is infinite and torsion, we conclude that $Z_p^n \leq G$, for each positive integer n .

We claim that, G is a divisible group.

Let n, m be two positive integers.

Assume $n = p^{\alpha_1} p_1^{\alpha_1} \dots p_k^{\alpha_k}$, where $p_1, \dots, p_k$ are distinct primes and $\alpha, \alpha_1, \dots, \alpha_k \geq 0$.

Since $n Z_p^{n+m} \cong Z_p^m$,

So $nG = G$ and the claim is proved.

Thus, by theorem, $G \cong Z(P^\infty)$.

Since, there is a unique subgroup of order p .

The other side is clear.

Hence the proof.

Bibliography



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[1] B.Csakany and G.Pollak,

The graph of subgroups of a finite group . (Russian) ,Czechoslovak math.J.19(1969) 241-247.

[2] Chakrabarty,S.Ghosh., T.K.Mukherjee and M.Sen,

Intersection graph of ideals of rings Discrete Math. 309(2009) 5381-5392

[3] W.Feit and J.G.Thompson,

Solvability of groups of odd order, pacific j. Math. 13(1963) 775-1029.

[4] P.Hall, A note on soluble groups, J. London Math.soc. 3(1928) 98-105

W.R. scott, Group Theory (Prentice- Hall, 1964).

[5] R.Shen, Intersection graphs of subgroups of finite groups.

Czech Math. J .60(2010) 945-950.

[6] B. Zelinka, Intersection graphs of finite abelian groups

Czech Math. J .25(1975)171-174.

[7] S.Akbari.H.A.Tavallace and S.Khalashi Ghezelahind,

Intersection graph of submodules of a module.J. Algebra Appl. 11(2012)

Article No.1250019

[8] J. Bosak, The graphs of semigroups, in theory of Graphs and Application

(Academic Press, New York, 1964), pp, 119-125.

[9]S. Akbari, R.Nikandish and M.J. Nikmehr, Some results on the intersection

Graphs of ideals of rings , J .Algebra Appl.12(2013)

Article No. 1250200.