



SECOND HANKEL DETERMINANT FOR A NEW SUBCLASS OF MULTIVALENT FUNCTION DEFINED BY A LINEAR DIFFERENTIAL OPERATOR

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Abstract

The main object of this paper is obtained a new sharp upper bound of second Hankel determinant for the subclass of univalent function $S_p^{m,n,b,\delta}(\phi)$ in the open unit disk U which is defined by Al-Oboudi Differential Operator.

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Analytic function, Univalent function, Starlike and convex function, Subordination, Second Hankel determinant and Al-Oboudi differential operator.

Introduction and Definitions

Definition 1.1

Let A denote the class of functions f normalized by

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \quad (1.1)$$

Which is analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

Definition 1.2

Let A_p denote the class of functions f normalized by $f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j}$ (1.2)



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Which is analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

In 1976, Noonan and Thomas [11] define the q th Hankel determinant of f for $q \geq 1$ & $n \geq 1$, which is defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdot & \cdot & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdot & \cdot & a_{n+q} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+q-1} & a_{n+q} & \cdot & \cdot & a_{n+2q-2} \end{vmatrix} \quad (1.3)$$

The Hankel determinant $H_q(n)$ has been investigated by several authors to study its rate of growth as $n \rightarrow \infty$ and to determine the bounds on it for specific values of q & n .

For example, Pommerenke [3] studied Hankel determinant of univalent function satisfy $|H_q(n)| < Kn^{-(\frac{1}{2} + \beta)q + \frac{3}{2}}$.

Later, Hayman [17] studied $|H_q(n)| < An^{\frac{1}{2}}$, Noor studied Hankel determinant for Bazilevic functions and close to convex function Noor [9], the Fekete and Szego [10] considered the second Hankel determinant $H_2(1) = a_3 - a_2^2$ for univalent functions, Janteng [1] investigated the sharp upper bound for second Hankel determinant $|H_2(2)| = |a_2 a_4 - a_3^2|$ for univalent functions and D. Bansal [4] obtained second Hankel determinant for some new class of analytic function.

Recently, Lee et al. [13] have studied and obtained bound on $|H_2(2)| = |a_2 a_4 - a_3^2|$ for the subclass of Ma-Minda starlike and convex functions. By this brief study, we investigate and obtained sharp upper bound of second Hankel determinant for some new class $S^{m,n,\delta,b}(\phi)$ of analytic functions.

Definition 1.3

Let A_p denote the class of function of $f(z)$ of p -valent analytic function

$$f(z) = z^p + \sum_{j=1}^{\infty} a_{p+j} z^{p+j} \quad (1.4)$$

Which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.

For $f \in A_p$, Al-Oboudi [1] introduces the following operator.

$$D^0 f(z) = f(z), \quad (1.5)$$

$$D'f(z) = D f(z) = z^p + \sum_{j=1}^{\infty} \left(\frac{p + (j-p)\delta}{p} \right) a_{p+j} z^{p+j} = D_{\delta} f(z) \quad (1.6)$$



$$D^n f(z) = D_\delta (D^{n-1} f(z)), \quad n \in N = 1, 2, 3, \dots \quad (1.7)$$

$$\therefore D^n f(z) = z^p + \sum_{j=1}^{\infty} \left(\frac{p + (j-p)\delta}{p} \right)^n a_{p+j} z^{p+j}, \quad n \in N_0 = N \cup \{0\}.$$

(1.8)

If $\delta = 1$, then we get Salagean [7] differential operator.

Definition 1.4

Let $\phi(z)$ be an analytic function with positive real part on U with $\phi(0) = 1, \phi'(0) > 0$, which maps the unit disc U onto the region Starlike with respect to one which is symmetric with respect to the real axis.

Let $S_{b,p}^*(\phi)$ be the class of functions $f(z) \in A_p$ for which $1 + \frac{1}{b} \left(\frac{1}{p} \frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z), z \in U$ and

$C_{b,p}(\phi)$ be the class of functions $f(z) \in A_p$ for which $1 - \frac{1}{b} + \frac{1}{bp} \left(1 + \frac{zf''(z)}{f'(z)} \right) \prec \phi(z), z \in U$ the Classes

$S_{b,p}^*(\phi)$ and $C_{b,p}(\phi)$ were introduced and studied by Ali et al [12]

We note that

1. $S_{1,1}^*(\phi) = S^*(\phi)$ and $C_{1,1}(\phi) = C(\phi)$, Ma and Minda [18] introduced and studied the class $S^*(\phi)$

consist of function $f(z) \in A$ for which $\frac{zf'(z)}{f(z)} \prec \phi(z), z \in U$. And the class $C(\phi)$ consist of functions

$f(z) \in A$ for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), z \in U$.

2. $S_{b,1}^*(\phi) = S_b^*(\phi)$ and $C_{b,1}(\phi) = C_b(\phi)$, Ravichandran et al. [16] introduced and studied the class $S_b^*(\phi)$

consist of function $f(z) \in A$ for which $1 + \frac{1}{b} \left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \phi(z), z \in U$ and the class $C_b(\phi)$ consist of

functions $f(z) \in A$ for which $1 + \frac{1}{b} \left(\frac{zf''(z)}{f'(z)} \right) \prec \phi(z), z \in U$.

Definition 1.5

Let $S_p^{m,n,b,\delta}(\phi)$ denoted the sub class of A consisting of functions f which satisfy the inequality $\operatorname{Re} \left(1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) \right) > \alpha$ (1.9)

For some $0 \leq \alpha < 1$, $b \in C - \{0\}$, $m \in N$, $n \in N_0 = N \cup \{0\}$, and all $z \in U$.

Definition 1.6

Let $S_p^{m,n,b,\delta}(\phi)$ denoted the sub class of A consisting of functions f which satisfy the inequality $\operatorname{Re} \left(1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) \right) \prec \phi(z)$ (1.10)

For some $b \in C - \{0\}$, $m \in N$, $n \in N_0 = N \cup \{0\}$, and all $z \in U$.

Definition 1.7



Let two functions f and g are analytic in U , we say that $f(z)$ is subordinate to $g(z)$ if there exist a function $w(z)$ analytic in U satisfying $w(0)=0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. It is denoted by $f(z) \prec g(z)$, by Schwarz lemma.

2. Main Result

Lemma 2.1 (see Duren [5])

If $P(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots$ is a function with positive real part in U , then sharp estimate, holds.

Lemma 2.2 (see U. Grenander [14])

If $P(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \dots$ is a function with positive real part in U , then $2c_2 = c_1^2 + x(4 - c_1^2)$ for some $x, |x| \leq 1$, and $4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$, for some $z, |z| \leq 1$.

Theorem 2.3

Let $\phi(z) = 1 + B_1z + B_2z^2 + \dots$, with $B_1 \neq 0$. If $f \in A$ satisfy the inequality $1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) \prec \phi(z)$, then $\left| a_{p+1}a_{p+3} - a_{p+2}^2 \right| \leq \frac{b^2 B_1^2}{\left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2}$.

(2.1)

Then the result is sharp.

Proof

If $f \in S_p^{m,n,b,\delta}(\phi)$, then there is a Schwarz function $w(z)$, analytic in U with $w(0)=0$ and $|w(z)| < 1$ in U . such that $1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) = \phi(w(z))$

(2.2)

Define $P(z)$ by

$$P(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \dots \quad (2.3)$$

Since $w(z)$ is a Schwarz function, it is clear that $\operatorname{Re} P(z) > 0$ and $P(0) = 1$.

$$\begin{aligned} \therefore \phi(z) &= \phi\left(\frac{P(z)-1}{P(z)+1}\right) = 1 + \frac{B_1c_1}{2}z + \left[\frac{B_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_2c_1^2}{4} \right] z^2 \\ &\quad + \left[\frac{B_1}{2} \left(c_3 - c_1c_2 + \frac{c_1^3}{4} \right) + \frac{B_2c_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_3c_1^3}{8} \right] z^3 + \dots \end{aligned}$$

Now from (2.2)



$$1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) = 1 + \frac{B_1 c_1}{2} z + \left[\frac{B_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right] z^2 + \left[\frac{B_1}{2} \left(c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) + \frac{B_2 c_1}{2} \left(c_2 - \frac{c_1^2}{2} \right) + \frac{B_3 c_1^3}{8} \right] z^3 + \dots \quad (2.4)$$

Equating the like coefficients,

$$\therefore a_2 = \frac{b B_1 c_1}{2 \left[\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right]},$$

$$a_3 = \frac{b B_1 c_2}{2 \left[\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right]}$$

$$+ \frac{b B_1 c_1^2}{4 \left[\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right]} \left[\frac{\left[\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right] b B_1}{\left[\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right]^2} - \left(1 - \frac{B_2}{B_1} \right) \right]$$

and



$$\begin{aligned}
 a_4 = & \frac{b^2(B_1B_2 - B_1^2)c_1^3 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right)}{8 \left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \\
 & + \frac{b^3B_1^3c_1^3 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right) \left(\frac{p+\delta}{p} \right)^n}{8 \left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \\
 & - \frac{b^3B_1^3c_1^3 \left(\left(\frac{p+\delta}{p} \right)^{m+2n} - \left(\frac{p+\delta}{p} \right)^{3n} \right)}{8 \left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^3} \\
 & + \frac{b(B_3 - 2B_2 + B_1)c_1^3}{8 \left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right)} \\
 & + \frac{b^2B_1^2c_1c_2 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right)}{4 \left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \\
 & + \frac{b(B_2 - B_1)c_1c_2}{2 \left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right)} \\
 & + \frac{bB_1c_3}{2 \left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right)}
 \end{aligned}$$



$$\begin{aligned}
 & \left[\frac{24b^2 B_1}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \right] C_1 C_3 \\
 & + \left[\frac{12b^3 B_1^2 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \right. \\
 & + \frac{24b^2 (B_2 - B_1)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} - \frac{24b^2 (B_2 - B_1)}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} \left. \right] C_1^2 C_2 \\
 & - \frac{24b^3 B_1^2 (1+\delta)^n}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2 \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \\
 & + \left[- \frac{24b^2 B_1}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} \right] C_2^2 \\
 & a_2 a_4 - a_3^2 = \frac{B_1}{96} \left\{ \begin{aligned} & 6b^3 (B_1 B_2 - B_1^2) \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right) \\ & \frac{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2}{6b^4 B_1^3 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right) \left(\frac{p+\delta}{p} \right)^n} \\ & + \frac{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^3}{6b^4 B_1^3 \left(\left(\frac{p+\delta}{p} \right)^{m+2n} - \left(\frac{p+\delta}{p} \right)^{3n} \right)} \\ & + \frac{6b^2 (B_3 - 2B_2 + B_1)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} C_1^4 \\ & - \frac{6b^2 \left(\frac{B_2^2}{B_1} + B_1 - 2B_2 \right)}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} - \frac{6b^4 B_1^3 (1+\delta)^{2n}}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2 \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \\ & - \frac{12b^3 B_1 (B_2 - B_1) (1+\delta)^n}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2 \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \end{aligned} \right\}
 \end{aligned}$$



(2.5)

then

$$|a_2 a_4 - a_3^2| = T |d_1 c_1 c_3 + d_2 c_1^2 + d_3 c_2^2 + d_4 c_1^4| \quad (2.6)$$

where

$$d_1 = \left[\frac{24b^2 B_1}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \right],$$

$$d_2 = \left[\frac{12b^3 B_1^2 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \right. \\ \left. + \frac{24b^2 (B_2 - B_1)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} - \frac{24b^2 (B_2 - B_1)}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} \right. \\ \left. - \frac{24b^3 B_1^2 (1+\delta)^n}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2 \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \right],$$

$$d_3 = \left[- \frac{24b^2 B_1}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} \right]$$

and



$$d_4 = \left[\frac{6b^3(B_1B_2 - B_1^2) \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \right. \\ + \frac{6b^4B_1^3 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right) \left(\frac{p+\delta}{p} \right)^n}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^3} \\ - \frac{6b^4B_1^3 \left(\left(\frac{p+\delta}{p} \right)^{m+2n} - \left(\frac{p+\delta}{p} \right)^{3n} \right)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^4} + \frac{6b^2(B_3 - 2B_2 + B_1)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \\ - \frac{6b^2 \left(\frac{B_2^2}{B_1} + B_1 - 2B_2 \right)}{6b^4B_1^3(1+\delta)^{2n}} - \frac{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2 \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \\ - \frac{12b^3B_1(B_2 - B_1)(1+\delta)^n}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2 \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \left. \right]$$

and $T = \frac{B_1}{96}$.

Since we assume without loss of generality that $c_1 > 0$ ($c \in [0,2]$) and substituting the value of c_2 & c_3 respectively.

Using lemma (2.3) and (2.6),

$$|a_2a_4 - a_3^2| = \frac{T}{4} \left| c^4(d_1 + 2d_2 + d_3 + 4d_4) + 2xc^2(4 - c^2)(d_1 + d_2 + d_3) \right. \\ \left. + x^2(4 - c^2)(-d_1c^2 + d_3(4 - c^2)) + 2d_1c(4 - c^2)(1 - |x|^2)z \right| \quad (2.7)$$

Now replace $|x|$ by μ and substituting the values d_1, d_2, d_3 & d_4 , then we get



$$|a_2 a_4 - a_3^2| \leq \frac{T}{4}$$

$$\begin{aligned}
 & \left[\frac{24b^3 B_1 B_2 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \right. \\
 & \quad \left. - \frac{48b^3 B_1 B_2 \left(\frac{p+\delta}{p} \right)^n}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} + \frac{24b^2 B_3}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \right. \\
 & \quad \left. + \frac{24b^4 B_1^3 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right) \left(\frac{p+\delta}{p} \right)^n}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^3} \right. \\
 & \quad \left. + \frac{24b^4 B_1^3 \left(\left(\frac{p+\delta}{p} \right)^{m+2n} - \left(\frac{p+\delta}{p} \right)^{3n} \right)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^4} - \frac{24b^4 B_1^3 (1+\delta)^{2n}}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2 \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \right. \\
 & \quad \left. - \frac{24b^2 \frac{B_2^2}{B_1}}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} \right] \\
 & + 2\mu c^2 (4-c^2) \left[\frac{12b^3 B_1^2 \left(\left(\frac{p+2\delta}{p} \right)^m \left(\frac{p+\delta}{p} \right)^n - \left(\frac{p+2\delta}{p} \right)^n \left(\frac{p+\delta}{p} \right)^n + \left(\frac{p+\delta}{p} \right)^m \left(\frac{p+2\delta}{p} \right)^n - \left(\frac{p+\delta}{p} \right)^n \left(\frac{p+2\delta}{p} \right)^n \right)}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} \right. \\
 & \quad \left. + \frac{24b^2 B_2}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} - \frac{24b^2 B_2}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} \right. \\
 & \quad \left. - \frac{24b^3 B_1^2 (1+\delta)^n}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)} \right] \\
 & + \mu^2 (4-c^2) \left[\frac{24b^2 B_1 c^2}{\left(\left(\frac{p+3\delta}{p} \right)^m - \left(\frac{p+3\delta}{p} \right)^n \right) \left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right) \left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2} + \frac{24b^2 B_1 (4-c^2)}{\left(\left(\frac{p+2\delta}{p} \right)^m - \left(\frac{p+2\delta}{p} \right)^n \right)^2} \right]
 \end{aligned}$$



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$$\equiv F(c, \mu)$$

If $\frac{\partial F}{\partial \mu} > 0$, then $F(c, \mu)$ increasing function of μ .

Hence for fixed $c \in [0, 2]$ and $\mu = 1$, the maximum of $F(c, \mu)$ occurs and $F(c, \mu) \equiv F(c, 1) \equiv G(c)$ (say)

$$\left| a_{p+1} a_{p+3} - a_{p+2}^2 \right| \leq \frac{b^2 B_1^2}{\left(\left(\frac{p+\delta}{p} \right)^m - \left(\frac{p+\delta}{p} \right)^n \right)^2}.$$

Letting, $m=1, n=0, b=1, \delta=1$ and $p=1$ in theorem (2.3) reduces to See Keong Lee et. Al. [13]

Corollary 2.4

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$, with $B_1 \neq 0$. If $f \in A$ satisfy the inequality $1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) \prec \phi(z)$, then $|a_2 a_4 - a_3^2| \leq \frac{B_1^2}{4}$.

Letting, $m=1, n=0, b=1, \delta=1$ and $p=1$ in theorem (2.3) reduces to See Keong Lee et. Al. [13]

Corollary 2.5

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$, with $B_1 \neq 0$. If $f \in A$ satisfy the inequality $1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) \prec \phi(z)$, then $|a_2 a_4 - a_3^2| \leq \frac{B_1^2}{36}$.

Letting, $m=1, n=0, b=1, \delta=1, p=1$ & $B_1 = B_2 = B_3 = 2$ in theorem (2.3) reduces to A. Janteng et. al. [1]

Corollary 2.6

Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \dots$, with $B_1 \neq 0$. If $f \in A$ satisfy the inequality $1 + \frac{1}{b} \left(\frac{D^m f(z)}{D^n f(z)} - 1 \right) \prec \phi(z)$, then

(i). $|a_2 a_4 - a_3^2| \leq \frac{1}{16}$. (See J. Sokol et al. [8].

(ii). $|a_2 a_4 - a_3^2| \leq \frac{16}{\pi^4}$. (See F. Ronning et al. [6].

(iii). $|a_2 a_4 - a_3^2| \leq \beta^2$. (See R. M, Ali et al. [12].

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