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On Prime Distance Labeling of Non-Commuting Graph of Non-Abelian Symmetric Groups

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Abstract: In this paper, we investigate if prime distance labeling of non-commuting graph of non-abelian symmetric groups (S_n, o), $n \geq 3$, exists or not.

Keywords: Prime Distance Labeling, Non-Commuting Graph, Non-Abelian Group, Centre of Group, Symmetric Group

1. Introduction

All the groups discussed in this paper are finite. In 1975, Paul Erdos considered the non-commuting graph for the very first time. Later in 2014, Abdollahi et al. studied certain properties of the non-commuting graph of a group. For more results, one can see [1, 2]. We use standard notations as $\mathbb{Z}$ for the set of integers, $Z(G)$ for the centre of group $(G, *)$, and $\Gamma(G)$ for the non-commuting graph of group $(G, *)$. Let V and E be the vertex set and edge set of $\Gamma(G)$, respectively. First, we recall a few definitions.

Definition 1: [3]

A graph $G(V, E)$ is said to admit a prime distance labeling if there exists an one-to-one function $\varphi: V \rightarrow \mathbb{Z}$ such that $|\varphi(u) - \varphi(v)|$ is a prime number for every pair of adjacent vertices u and v in G.

For a detailed study on prime distance labeling of graphs one can refer to [4, 5, 6, 7, and 8].

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Definition 2:

A group $(G, *)$ is said to be non-abelian group if there exists at least two

elements a, b in G such that $a * b \neq b * a$.

Definition 3:

The centre of group $(G, *)$ is defined as the subset of G containing all those elements which commute with each other element of group G under binary operation $*$. It is denoted by $Z(G) = \{x \in G: x * g = g * x, \forall g \in G\}$.

Definition 4:

A non-commuting graph of any group $(G,*)$ is denoted by $\Gamma(G)$ having the vertex set $V = G - Z(G)$ and edge set is defined by $E = \{e : e$ is edge between vertex pair (u, v) such that $u * v \neq v * u$, for $u, v \in V\}$.

Definition 5:

The set of all permutations on finite set $X = \{1, 2, 3, \dots, n\}, n \in N$ forms a group with respect to operation *'o'* of composition of functions, known as symmetric group and denoted by (S_n, o), for $n \in N$.

2. Main Results

In this section, we prove that the prime distance labeling of non-commuting graph $\Gamma(G)$ does not exist for non-abelian symmetric groups $(S_n, o), n \geq 3$.

Theorem 2.1:

No triangular graph can possess prime distance labeling with all vertices labeled either odd integers or even integers.

Proof.

Consider a triangular graph with vertices v_1,v_2, and v_3 having prime distance labeling. Define a function φ: V → ℤ by $|\varphi(v_1) - \varphi(v_2)|, |\varphi(v_2) - \varphi(v_3)|, |\varphi(v\_1) - \varphi(v_3)|$ all are prime integers. (I)

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Suppose that each of φ (v_i) is an even integer, for $i = 1, 2$ and 3. Since, we know that difference of any two consecutive even integers is 2, which is odd prime and difference of any two non-consecutive even integers is $2n, n \geq 2$. So, if φ $(v_1) = 2n, n \in Z$, then $\varphi(v_2)$, $\varphi(v_3)$ have to be consecutive even integers to $f(v_1)$ i.e. φ $(v_2) = 2n + 2$ and $\varphi(v_3) = 2n - 2, n \in Z$. But this contradicts the statement (I) as $|\varphi(\mathrm{v}_2) - \varphi(\mathrm{v}_3)| = 4$, which is not a prime number. So, our supposition is wrong. Similarly, using the fact that difference of any two consecutive odd integers is 2, which is even prime and difference of any two non-consecutive odd integers is $2n, n \geq 2$, which is a composite integer. Therefore, one can see that no triangular graph can admit a prime distance labeling with all vertices either even or odd integers. So, if two vertices of a triangular graph are labeled with consecutive even (or odd) integers, then the remaining one should be an odd (or even) integer.

Theorem 2.2:

The non-commuting graph $\Gamma(S_3)$ of non-abelian symmetric group (S_3, o) does not permit a prime distance labeling.

Proof.

Symmetric group (S_3, o) consists of permutations on any three distinct objects, say 1, 2, 3. So, total number of permutations of these is 3! = 6. Hence, $S_3 = \{I, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ \ 3), (1\ 3\ 2)\}$, where I stands for identity element of group and operation *'o'* stands for operation of composition defined as below: Clearly, $Iof = f = foI$, for each f in S_3.

(1 2) o (1 3) = $\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\begin{pmatrix}1&3&2\\3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}$ = (1 3 2), (1 3) o (1 2) = $\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ = (1 2 3)

(2 3)o(1 3) = $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\begin{pmatrix}1&3&2\\3&1&2\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = (1\ 2\ 3)$, $(1\ 3)\ o\ (2\ 3) = \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}\begin{pmatrix}1&3&2\\1&2&3\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} =$ (1 3 2)

(1 2) o (2 3) = $\begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}\begin{pmatrix}1&3&2\\1&2&3\end{pmatrix} = \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = (1\ 2\ 3)$, (2 3) o (1 2) = $\begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}\begin{pmatrix}1&3&2\\2&3&1\end{pmatrix} = \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix} =$ (1 3 2)

(1 2 3) o (1 3 2) =$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}\begin{pmatrix}1&3&2\\3&2&1\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = I$, (1 3 2) o (1 2 3) = $\begin{pmatrix}1&3&2\\3&2&1\end{pmatrix}\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix} = \begin{pmatrix}1&2&3\\1&2&3\end{pmatrix} = I$

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So, it is clear that non-commuting elements of the symmetric group (S_3, o) are (1 2), (1 3), (1 2 3), (1 3 2), and (2 3). Therefore, the non-commuting graph $\Gamma(S_3)$ consists of vertex set V = {(1 2), (1 3), (1 2 3), (1 3 2), (2 3)} which can be written as V = $\{v_i : 1 \leq i \leq 5\}$ & the edge set $E = \{e_{ij} = (v_i, v_j) : v_i * v_j \neq v_j * v_i, 1 \leq i, j \leq 5\} = \{(v_1, v_2), (v_1, v_3), (v_1, v_4), (v_1, v_5), (v_2, v_3), (v_2, v_4), (v_2, v_5), (v_3, v_4), (v_3, v_5)\}$. Hence, one can get the non-commuting graph $\Gamma(S_3)$ as shown below in Figure 1.

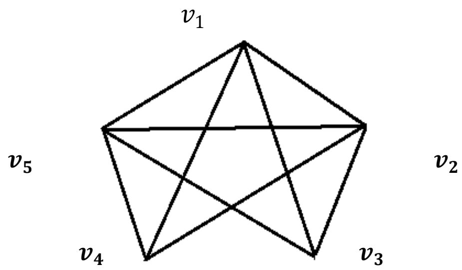

Figure 1. Non-Commuting Graph $\Gamma(S_3)$

It is important to note in Figure 1 that only the vertices v_3 & v_4 are not adjacent. Assume that the graph in Figure 1 admits a prime distance labeling which imply there exists a function $\varphi : V(\Gamma(S_3)) \to \mathbb{Z}$, such that $|\varphi(v_i) - \varphi(v_j)|$ is a prime number, for each $(v_i, v_j) = e_{ij}$ in E. Consider $\Delta v_1 v_2 v_3$, by Theorem 2.1, all vertices of $\Delta v_1 v_2 v_3$ cannot be labeled with only even integers or odd integers. So, let's assume that $\varphi(v_1)$ be even integer and $\varphi(v_3)$ be odd i.e. $\varphi(v_1) = 2n$, $\varphi(v_3) = 2m+1$, where $n, m \in Z$. Consequently, $\varphi(v_2)$ is either odd or even integer.

<u>**Case (i)**</u> : **When $\varphi(v_2)$ is an even integer**

Since (v_2, v_1) is pair of adjacent vertices in graph and graph admits a prime distance labeling, So, $\varphi(v_2) = 2n + 2$. Hence, by Theorem 2.1., in $\Delta v_1 v_2 v_5$, $\varphi(v_5)$ must be an odd integer and (v_3, v_5) being a pair of adjacent vertices implies that $\varphi(v_5) = 2m+3$. Similarly, $\Delta v_1 v_2 v_4$ implies $\varphi(v_4) = 2m + 5$. Since function φ defines prime distance labeling on graph $\Gamma(S_3)$ as shown in Figure 1 and using fact that the difference of odd and even integer is always odd intger, we have following interpretations:

(v_i, v_j) $\to$	(v_1, v_3)	(v_1, v_5)	(v_1, v_4)	(v_2, v_3)	(v_2, v_5)	(v_2, v_4)
$\mid \varphi(v_i)$-	P_1=	P_2=	P_3=	P_4=	P_1=	P_2=

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$\varphi(v_j)$|	|2(n-m)-1|	|2(n-m)-3|	|2(n-m)-5|	|2(n-m)+1|	|2(n-m)-1|	|2(n-m)-3|

Each number P_i (i = 1, 2, 3, and 4) in the second row must be an odd prime number. Let $2(n-m) = x$ and so, x is an even integer. Hence, our purpose is to find that even integer x such that each of $P_1 = |x-1|$, $P_2 = |x-3|$, $P_3 = |x-5|$, & $P_4 = |x+1|$ is an odd prime. Clearly, $x \neq 0, \pm 2, \pm 4$, otherwise either P_1 or P_2 or P_3 or P_4 will be equal to 1, which is not a prime number. (II)

Hence, we have only two possibilities either $x < -4$ or $x > 4$. In other words, we can say all $P_i's$ are of same sign. Without loss of generality, we assume $x > 4$, hence, $P_1 = x - 1$, $P_2 = x - 3$, $P_3 = x - 5$, $P_4 = x + 1$

$\Rightarrow P_1 - P_2 = 2, P_2 - P_3 = 2, P_4 - P_1 = 2$
$\Rightarrow (P_1, P_2), (P_2, P_3), (P_1, P_4)$ all are pair of twin odd primes. (III)

Also, it is important to note that (i) No P_i can be 3, (ii) None of P_2 and P_1 can be 5. (IV)

Because if $P_1 = 3$, then $x = 4$ which is contradiction to (II). If $P_2 = 3$, then $x = 6 \Rightarrow P_3 = 1$, which is impossible as P_3 is an odd prime. If $P_3 = 3$, then $x = 8 \Rightarrow P_4 = 9$, which is also impossible as P_4 is an odd prime. If $P_4 = 3$, then $x = 4$, which is a contradiction to (II). Further if $P_2 = 5$, then $x = 8 \Rightarrow P_4 = 9$, which is also impossible as P_4 is an odd prime. And if $P_4 = 5$, then $x = 4$, which is contradiiction to (II). So, assertion (IV) is true that (i) No P_i can be 3, (ii) None of P_2 and P_1 can be 5. Since every odd prime is of form $4k + 1$ or $4k + 3$, k is any integer. So, if $P_1 = 4k + 3$, then $P_2 = 4k + 1, P_3 = 4k - 1, P_4 = 4k + 5$. Or if $P_1 = 4k + 1$, then $P_2 = 4k - 1, P_3 = 4k - 3, P_4 = 4k + 3$, each of them is odd prime. Therefore, all $4k - 1, 4k + 1, 4k + 3, 4k + 5$ or $4k - 3, 4k - 1, 4k + 1$, and $4k + 3$ are consecutive odd primes for a fixed integer k. But we know that every third odd number is divisible by 3, which means that "no three successive odd numbers can be prime unless one of them is 3", which is impossible by (IV) as none of P_1, P_2, P_3, and P_4 is 3. Further (III) implies that odd prime P_2 is part of two twin primes pairs $(P_1, P_2), (P_2, P_3)$, which is possible only when $P_2 = 5$ and it contradicts assertion (IV). Hence, our supposition is wrong. So, $\varphi(v_2)$ cannot be an even integer. This case is rejected.

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Case (II): When φ (v_2) is an odd integer

In $\Delta v_1v_2v_3$, (v_2,v_3) is pair of adjacent vertices in graph and graph admits a prime distance labeling. So, by Theorem 2.1, in $\Delta v_1v_2v_3$, we get φ (v_2)= 2m+3, φ (v_1) = 2n, φ (v_3)= 2m+1, for n, m integers. Proceeding as in Case (i), we get the following labeling of graph in Figure 1 in this case, φ (v_5) = 2n+2, φ (v_4) = 2m+5, φ (v_2)= 2m+3, φ (v_1)= 2n, φ (v_3)= 2m+1. Since the function φ defines the prime distance labeling on graph as shown in Figure 1 and using the fact that the difference of odd and even integer is always an odd intger, we have the following interpretations:

	(v_1,v_3)	(v_1,v_4)	(v_1,v_2)	(v_5,v_3)	(v_5,v_4)	(v_5,v_2)
(v_i,v_j) →						
| φ (v_i)- φ (v_j)|	P_1= |2(n-m)-1|	P_2= |2(n-m)-5|	P_3= |2(n-m)-3|	P_4= |2(n-m)+1|	P_3= |2(n-m)-3|	P_1= |2(n-m)-1|

Each number P_i $(i = 1, 2, 3, \text{and } 4)$ in the second row must be an odd prime number. Let $2(n - m) = x$ and x is an even integer. Hence, our purpose is to find that even integer x such that each of $P_1 = |x - 1|$, $P_2 = |x - 5|$, $P_3 = |x - 3|$ & $P_4 = |x + 1|$ is an odd prime. This condition is similar to the one we obtained in Case (i), which imply that there doesn't exist any even integer x for which each P_i $(i = 1, 2, 3, 4)$, as deined above, is an odd prime. Hence, f(v_2) cannot be an odd integer and this case is rejected. So, we observe that φ(v_2) can be neither odd nor an even integer. Hence, there doesn't exist any function $\varphi : V(\Gamma(S_3)) \to \mathbb{Z}$, such that $|\varphi(v_i) - \varphi(v_j)|$ is a prime number, for each $(v_i, v_j) = e_{ij}$ in E, which simply clears that the non-commuting graph Γ(S_3) doesn't admit a prime distance labeling.

Theorem 2.3:

The non-commuting graph $\Gamma(S_n)$, $n \geq 4$, of non-abelian symmetric group (S_n, o), $n \geq 4$, does not permit a prime distance labeling.

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Proof.

Denote the non-commuting graph of group (S_n, o), n ≥ 4 by $\Gamma(S_n)$ whose vertex set consists of those elements of group (S_n, o) which don't commute with at least one element of group by V and the edge set of graph by E. So, V = $S_n - Z(S_n)$, where $Z(S_n)$ represents the centre of group S_n. Let if possible there exists a function $\varphi: V(\Gamma(S_n)) \to Z$ such that $|\varphi(u) - \varphi(v)|$ is a prime number, for each pair of adjacent vertices (u, v) in V. By the definition of composition function on group (S_n, o), n ≥ 4, we have the following results:

(12) o (23) =$\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}$, (23)o(12)= $\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\3 & 1 & 2\end{pmatrix}$

(12)o(24)= $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 4\\2 & 4 & 1\end{pmatrix}$, (24)o(12)= $\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 4\\4 & 1 & 2\end{pmatrix}$

(12)o(234)= $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 1\end{pmatrix}$

(234)o(12)= $\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 1 & 4 & 2\end{pmatrix}$

(12)o(123) = $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\1 & 3 & 2\end{pmatrix}$

(123)o(12) =$\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}$

(12)o(134)= $\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 1 & 4 & 2\end{pmatrix}$

(134)o(12)= $\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\2 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 1\end{pmatrix}$

(24)o(123)= $\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\4 & 3 & 1 & 2\end{pmatrix}$

(123)o(24)= $\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 1 & 3\end{pmatrix}$

(24)o(134)= $\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 2 & 1\end{pmatrix}$

(134)o(24)= $\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix}\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\3 & 1 & 4 & 2\end{pmatrix}$

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(24)o(234)= $\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\3 & 2 & 4\end{pmatrix}$

(234)o(24)= $\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\2 & 4 & 3\end{pmatrix}$

(24)o(23)= $\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}$, (23)o(24)= $\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\begin{pmatrix}2 & 4\\4 & 2\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\4 & 2 & 3\end{pmatrix}$

(23)o(123)= $\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\end{pmatrix}$

(123)o(23)= $\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 3\\2 & 1 & 3\end{pmatrix}$

(23)o(134)= $\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 & 4\\2 & 3 & 4 & 1\end{pmatrix}$

(134)o(23)= $\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix}\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 2 & 1\end{pmatrix}$

(23)o(234) = $\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\2 & 4 & 3\end{pmatrix}$

(234)o(23)= $\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}\begin{pmatrix}2 & 3\\3 & 2\end{pmatrix}=\begin{pmatrix}2 & 3 & 4\\4 & 3 & 2\end{pmatrix}$

(123)o(134)= $\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 & 4\\1 & 3 & 4 & 2\end{pmatrix}$

(134)o(123) = $\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 & 4\\2 & 4 & 3 & 1\end{pmatrix}$

(123)o(234)= $\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 & 4\\2 & 1 & 4 & 3\end{pmatrix}$

(234)o(123)= $\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\2 & 3 & 1\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 1 & 2\end{pmatrix}$

(134)o(234)= $\begin{pmatrix}1 & 3 & 4\\3 & 4 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 4\\3 & 4 & 2\end{pmatrix}=\begin{pmatrix}1 & 2 & 3 & 4\\3 & 4 & 1 & 2\end{pmatrix}$

From above, it is clear that at least (12), (23), (24), (123), (234), and (134) do not belong to the centre of group (S_n, o), for each n ≥ 4. Hence if S = {(12), (23), (24), (123), (234), (134)}, then $S \subseteq V$, then each of element of the set S doesn't commute with each of remaining elements of the set S. It implies that in the non-commuting graph $\Gamma(S_n)$ of group (S_n, o) for n ≥ 4, at least the following vertices are adjacent: (12)(23), (12)(24), (12)(123),

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(12)(234), (12)(134), (23)(24), (23)(123), (23)(234), (23)(134), (24)(123), (24)(134), (24)234), (123)(234), (123)(134), and (234)(134). Note that the vertex set V of $\Gamma(S_n)$ may have more elements, apart from the ones written above, that are adjacent to each other. But these are the minimum number of vertices which are adjacent in each $\Gamma(S_n)$ for n ≥ 4. Hence, the following diagram represents a sub-graph of graph $\Gamma(S_n)$ in which each extended line at vertex v_i shows that the corresponding vertex might be adjacent to other vertices too.

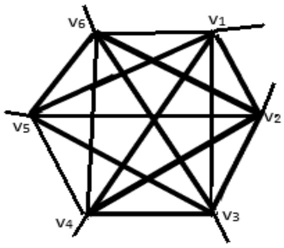

Figure 2. The Sub-graph of $\Gamma(S_n)$

Since function $\varphi: V(\Gamma(S_n)) \rightarrow Z$ determines the prime distance labeling on $\Gamma(S_n)$ such that

$|\varphi(v_i) - \varphi(v_j)|$ is a prime number, for each pair of adjacent vertices (v_i, v_j). In $\triangle\, v_1v_2v_3$, as by result of Theorem 2.1, all of three vertices can neither be odd nor be even simultaneously. So, let's assume that $\varphi(v_1)$ be an even integer and $\varphi(v_3)$ be odd i.e. φ (v_1)= 2n, φ (v_3)=2m+1, where $n, m \in Z$. Consequently, φ (v_2) is either an odd or even integer.

Case (i): When φ (v_2) is an even integer

Since (v_1, v_2) is a pair of adjacent vertices in Figure 2 and φ (v_1)= 2n, so in order for $|\varphi(v_1) - \varphi(v_2)|$ to be a prime number, $\varphi(v_2) = 2n + 2$. Also $\triangle\, v_1v_2v_6$ implies if, $\varphi(v_1)$ and $\varphi(v_2)$ are even integers, then $\varphi(v_6)$ must be an odd integer and (v_6, v_3) being an adjacent pair, by Theorem 2.1, it is easy to see that $\varphi(v_6)$ = 2m+3. Applying the same procedure to $\triangle\, v_6v_3v_4$, we get that $\varphi(v_4)$ must be an even integer and hence, $\varphi(v_4)$ = 2n-2, but (v_2, v_4) is a pair of adjacent vertices. So, $|\,\varphi(v_2) - \varphi(v_4)| = 4$, which is not a prime number. Therefore, we arrive at a contradiction and this case is rejected.

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Case(ii): When $\varphi(v_2)$ is an odd integer

Since (v_2, v_3) is a pair of adjacent vertices and φ (v_3)=2m+1, so $\varphi(v_2) = 2m + 3$. From $\triangle\, v_3v_2v_6, \varphi(v_6)$ must be an even integer and (v_1, v_6) being adjacent vertices, $\varphi(v_6) = 2n + 2$. Conseqently, $\triangle\, v_1v_4v_6$ implies that $\varphi(v_4)$ is an odd integer. Hence, in $\triangle\, v_2v_4v_3$, all vertices are labeled as odd integers, which is a contradiction as per Theorem 2.1. Therefore, this case is also rejected.

Hence, we conclude that $\varphi(v_2)$ can neither be an odd integer nor even integer which imply that the prime distnace labeling of a sub-graph of $\Gamma(S_n)$ as shown in Figure 2 is not possible. It is now obvious that the graph $\Gamma(S_n)$ doesn't admit a prime distance labeling by using the fact that if a graph does not permit a prime distance labeling then so does its supergraph. And so, one can conclude that the non-commuting graph of symmetric group (S_n, o), n $\geq$ 4 and n is a natural number, doesn't admit a prime distance labeling.

3. Conclusion

The non-existence of a prime distance labeling of the non-commuting graph of non-abelian symmetric groups (S_n, o), n $\geq$ 3 is established.

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