

Haar (Scale 3) Wavelet Based Solution of 1D- Hyperbolic Telegraph Equation.

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Abstract. The main focus of the current work is to introduce the new numerical technique based upon the Haar scale 3 wavelets for the solution of 1D- hyperbolic telegraph equation. In this scheme, spatial discretization is done by scale 3 Haar wavelets. The collocation method is used with Haar scale 3 wavelets to convert 1D- hyperbolic telegraph equation into the system of algebraic equations which are further solved with the help of Thomas algorithm. The proposed scheme is tested on four different equations of above said types to establish the competency of the proposed scheme.

Keywords: Scale 3 Haar wavelets, 1D- hyperbolic telegraph equation, Partial differential Equations (65L10)

1. Introduction

The prime objective of current work is to provide new numerical technique for obtaining the solution of second order hyperbolic Telegraph equation of the following type, using two-dimensional Haar Scale 3 wavelet bases.

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\alpha \frac{\partial \varphi}{\partial t} + \beta^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + g(x, t), (x, t) \in [0, 1] \times [0, T] \quad (1.1)$$

under the following types of initial constraints

$$\varphi(x, 0) = \psi_1(x)$$

$$\frac{\partial \varphi}{\partial t}(x, 0) = \psi_2(x)$$

and the boundary constraints

$$\varphi(0, t) = \xi_1(t), \quad t \in [0, T]$$

$$\varphi(1, t) = \xi_2(t), \quad t \in [0, T]$$

Where $\alpha > \beta \geq 0$ are known constants, $\psi_1(x), \psi_2(x), \xi_1(t), \xi_2(t), g(x, t)$ are the given functions and $\varphi(x, t)$ is the function whose value is to be determined. When $\alpha > 0, \beta = 0$ then equation represents the damped wave motion equation and when $\alpha > \beta > 0$ then it is named as the telegraph Equation. These types of equations emerging in the field of electric signal propagation in cables[1], pulsate blood flow in the arteries[2], 'acoustic wave propagation' in porous media of Darcy-type[3], vibrations in the different structures,

walktheory[4], Maxwell viscous fluid parallel flow study[5], etc. Telegraph equation is a special type of hyperbolic equation which is more suitable in modelling the reaction-diffusion as compared to the ordinary diffusion equation for many branches of engineering and sciences. Telegraph equation plays a very crucial role in the area of atomic physics and it is considered as one of the fundamental equations of atomic physics. Finding the solution of these types of equations becomes a big challenge as there is no established analytic method to solve these kinds of problems. Therefore, many researchers are involved in developing the various numerical and semi-analytic schemes for finding solutions to the different problems governed by telegraph equations. Some of the methods which have been freshly developed and applied to solve the telegraph equations are Septic B-Spline Collocation Method (SBSCM) [6], Cubic B-Spline Quasi-Interpolation Method (CBSQIM) [7], Cubic B-Spline Collocation Method (CBSCM) [8]–[10], Reproducing Kernel Hilbert Space Method (RKHSM) [11], Radial Basis Function Method (RBFM) [12], Dual Reciprocity Boundary Integral Equation Method (DRBIEM) [12], Differential Quadrature Method (DQM) [13], Trigonometric B-Splines Method (TBSM) [9], Homotopy Analysis Method (HAM) [14], Chebyshev Tau Method (CTM) [15], etc.

But the study of establishing the solution for Telegraph equation using the Haar scale 3 wavelets has not been attempted in the literature which motivated us to develop a new technique for the solution of these types of equations. Wavelets are one of the modernistic orthonormal functions which have a capability of dilation and translation. Because of these properties, numerical techniques which involve wavelet bases are showing qualitative improvement in contrast with other methods. In literature, dyadic wavelets are in preponderance. In 1995, Chui and Lian [16] has developed the Haar scale 3 (non-dyadic) wavelets by using the process of multiresolution analysis. In 2018, Mittal and Pandit have used the Haar scale 3 wavelets [17]–[20] for solving the various types of differential equations and found that these wavelet bases are equally competent in solving the various types of mathematical models governed by differential equations. Also, it was shown by them that the Haar scale 3 wavelet has a faster rate of convergence as compared to the dyadic wavelets. This gives us good hope of getting a better solver for these equations by developing a new hybrid technique based upon Haar scale 3 Wavelets for the solution of Telegraph equations.

The manuscript follows the sequence of sections as described: In section 2, explicit forms of Haar scale 3 parent wavelets with their families and procedure to find their integrals have been explained briefly. Representation of the solution using Haar scale 3 wavelets is explained in section 3. Section 4 explains the method of solution using Haar scale 3 wavelets. In section 5, solutions of three different Telegraph equations are produced to analyze the competence and performance of the method. In section 6, the conclusion drawn from the results and in future research idea is given.

2. Explicit forms of Haar scale 3 Wavelets and their integrals

The mathematical expressions for Haar scale 3 wavelet family [16], [19] are given below

$$\mathcal{H}_1(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{elsewhere} \end{cases} \quad (2.2)$$

$$\mathcal{H}_i(t) = \psi^1(3^j t - k) = \frac{1}{\sqrt{2}} \begin{cases} -1 & \alpha_1(i) \leq t < \alpha_2(i) \\ 2 & \alpha_2(i) \leq t < \alpha_3(i) \\ -1 & \alpha_3(i) \leq t < \alpha_4(i) \\ 0 & \text{elsewhere} \end{cases}, \quad \text{for } i = 2, 4, \dots, 3p-1 \quad (2.3)$$

$$\mathcal{H}_i(t) = \psi^2(3^j t - k) = \frac{1}{\sqrt{2}} \begin{cases} 1 & \alpha_1(i) \leq t < \alpha_2(i) \\ 0 & \alpha_2(i) \leq t < \alpha_3(i) \\ -1 & \alpha_3(i) \leq t < \alpha_4(i) \\ 0 & \text{elsewhere} \end{cases}, \quad \text{for } i = 3, 6, \dots, 3p \quad (2.4)$$

where $\alpha_1(i) = \frac{k}{p}$, $\alpha_2(i) = \frac{3k+1}{3p}$, $\alpha_3(i) = \frac{(3k+2)}{3p}$, $\alpha_4(i) = \frac{k+1}{p}$, $p = 3^j$, $j = 0, 1, 2, \dots$, $k = 0, 1, 2, \dots, p-1$.

Here i, j, k respectively represent the wavelet number, level of resolution (dilation) and translation parameters of wavelets family. The values of i (for $i > 1$) can be calculated with the help of j, k by using the following relations $\begin{cases} i-1 = 3^j + 2k & \text{for even } i \\ i-2 = 3^j + 2k & \text{for odd } i \end{cases}$. By using this relation for different dilation and translations of $\mathcal{H}_2(t), \mathcal{H}_3(t)$, we will get the wavelet family where $\mathcal{H}_2(t)$ and $\mathcal{H}_3(t)$ are also called mother wavelets and rest all the wavelets which we have obtained from mother wavelet are called daughter wavelets.

The main difference which makes the Haar Scale 3 wavelets better than the dyadic wavelets is that only one mother wavelet is responsible for the construction of whole wavelet family but in case of Haar Scale 3 wavelets, two mother wavelets with different shapes are responsible for the construction of the whole family. Because of this fact, Haar Scale 3 wavelets increase the convergence rate of the solution.

Now one can easily integrate the equations [Eq. (2.1)-Eq. (2.3)] the desired number of times over the interval $[A, B]$ by using Riemann Liouville Integral formula as given below

$$\mathcal{Q}_{\beta,i}(t) = \frac{1}{\Gamma(\beta)} \int_A^t h_i(x) (t-x)^{\beta-1} dx; \quad 0 \leq \beta \leq m, \quad m = 1, 2, 3, \dots, \quad i = 1, 2, 3, \dots, 3p \quad (2.5)$$

After evaluating the above integrals, we get

$$\mathcal{Q}_{\beta,i}(t) = \frac{t^\beta}{\Gamma(\beta+1)} \quad \text{for } i = 1 \quad (2.6)$$

$$\mathcal{Q}_{\beta,i}(t)'s \text{ for } i = 2, 4, 6, 8, \dots, 3p-1 \text{ are given below } \mathcal{Q}_{\beta,i}(t) = \left. \begin{aligned} & 0 && \text{for } 0 \leq t \leq \alpha_1(i) \\ & \frac{-1}{\Gamma(\beta+1)} (t - \alpha_1(i))^\beta && \text{for } \alpha_1(i) \leq t \leq \alpha_2(i) \\ & \frac{1}{\Gamma(\beta+1)} [-(t - \alpha_1(i))^\beta + 3(t - \alpha_2(i))^\beta] && \text{for } \alpha_2(i) \leq t \leq \alpha_3(i) \\ & \frac{1}{\Gamma(\beta+1)} [-(t - \alpha_1(i))^\beta + 3(t - \alpha_2(i))^\beta - 3(t - \alpha_3(i))^\beta] && \text{for } \alpha_3(i) \leq t \leq \alpha_4(i) \\ & \frac{1}{\Gamma(\beta+1)} [-(t - \alpha_1(i))^\beta + 3(t - \alpha_2(i))^\beta - 3(t - \alpha_3(i))^\beta + (t - \alpha_4(i))^\beta] && \text{for } \alpha_4(i) \leq t \leq 1 \end{aligned} \right\} \quad (2.7)$$

$$Q_{\beta,i}(t)'s \text{ for } i = 3,5,7,9,\dots,3p \text{ are given by } Q_{\beta,i}(t) = \left. \begin{array}{ll} 0 & \text{for } 0 \leq t \leq \alpha_1(i) \\ \frac{1}{\Gamma(\beta+1)}(t-\alpha_1(i))^\beta & \text{for } \alpha_1(i) \leq t \leq \alpha_2(i) \\ \sqrt{\frac{3}{2}} \left\{ \begin{array}{ll} \frac{1}{\Gamma(\beta+1)}[(t-\alpha_1(i))^\beta - (t-\alpha_2(i))^\beta] & \text{for } \alpha_2(i) \leq t \leq \alpha_3(i) \\ \frac{1}{\Gamma(\beta+1)}[(t-\alpha_1(i))^\beta - (t-\alpha_2(i))^\beta - (t-\alpha_3(i))^\beta] & \text{for } \alpha_3(i) \leq t \leq \alpha_4(i) \\ \frac{1}{\Gamma(\beta+1)}[(t-\alpha_1(i))^\beta - (t-\alpha_2(i))^\beta - (t-\alpha_3(i))^\beta + (t-\alpha_4(i))^\beta] & \text{for } \alpha_4(i) \leq t \leq 1 \end{array} \right\} \end{array} \right\} \quad (2.8)$$

3. Approximation of solution

Using the properties of Haar Scale 3 wavelets as explained in section 3, any function $u(x, t) \in L_2(R)$ can be approximated as

$$u(x, t) = \sum_{i=1}^{\infty} \sum_{l=1}^{\infty} a_{il} \mathcal{H}_i(x) \mathcal{H}_l(t) \quad (3.9)$$

Where a_{il} 's will be determined by the proposed method. But for the computational purpose, one can consider a truncated series up to $3p \times 3p$ terms. By considering the $3p \times 3p$ terms to approximate the function $u(x, t)$ we get

$$u(x, t) \approx u_{3p}(x, t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} \mathcal{H}_i(x) \mathcal{H}_l(t) \quad (3.10)$$

where $p = 3^j, j, j' = 0, 1, 2, \dots$

4. Method of Solution

Now space variables are discretized with the help of 'Haar scale 3 wavelets' as explained below

$$\varphi_{xxtt}(x, t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} \mathcal{H}_i(x) \mathcal{H}_l(t) \quad (4.11)$$

Integrating w.r.t x within the domain 0 to x , the above equation leads to

$$\varphi_{xtt}(x, t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} Q_{1,i}(x) \mathcal{H}_l(t) + \varphi_{xtt}(0, t) \quad (4.12)$$

Now by integrating the above equation w.r.t x within the domain 0 to 1, the value of $\mathcal{U}_{xtt}(0, t)$ is given by

$$\varphi_{xtt}(0, t) = (\varphi_{tt}(1, t) - \varphi_{tt}(0, t)) - \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} Q_{2,i}(1) \mathcal{H}_l(t) \quad (4.13)$$

Now equation becomes

$$\varphi_{xtt}(x, t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} \left(Q_{1,i}(x) - Q_{2,i}(1) \right) \mathcal{H}_l(t) + (\varphi_{tt}(1, t) - \varphi_{tt}(0, t)) \quad (4.14)$$

Again, integrating w.r.t x within the limits we get

$$\varphi_{tt}(x, t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} (Q_{2,i}(x) - x Q_{2,i}(1)) \mathcal{H}_l(t) + x \varphi_{tt}(1, t) + (1 - x) \varphi_{tt}(0, t) \quad (4.15)$$

Taking the limit 0 to t , integrating w.r.t t and then by applying the boundary conditions, we get

$$\varphi_t(x, t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} (Q_{2,i}(x) - x Q_{2,i}(1)) Q_{1,l}(t) + x (\varphi_t(1, t) - \psi_2(1)) \quad (4.16)$$

Integrate w.r.t t within the same limit implies

$$\varphi(x, t) = \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} (Q_{2,i}(x) - x Q_{2,i}(1)) Q_{2,l}(t) + x (\xi_2(t) - \xi_2(0)) - x t \psi_2(1) + (1 - x) (\xi_1(t) - \xi_1(0)) - (1 - x) t \psi_2(0) \quad (4.17)$$

Differentiating the above equation two times with respect we get φ_{xx} , can be obtained as

$$\begin{aligned} \varphi_{xx}(x, t) &= \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} \mathcal{H}_i(x) Q_{2,l}(t) + t(\psi_2)_{xx}(x) \\ \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} (Q_{2,i}(x) - x Q_{2,i}(1)) \mathcal{H}_l(t) &+ x (\xi_2(t))_{tt} + (1 - x) (\xi_1(t))_{tt} \\ &+ \alpha \left(\sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} (Q_{2,i}(x) - x Q_{2,i}(1)) Q_{1,l}(t) + x (\varphi_t(1, t) - \psi_2(1)) \right. \\ &\left. + (1 - x) (\varphi_t(0, t) - \psi_2(0)) + \psi_2(x) \right) \end{aligned} \quad (4.18)$$

$$\begin{aligned} &\sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} \left[(Q_{2,i}(x) - x Q_{2,i}(1)) \mathcal{H}_l(t) + \right. \\ &\alpha Q_{2,i}(x) - x Q_{2,i}(1) Q_{1,l}(t) + \beta Q_{2,i} x - x Q_{2,i} Q_{2,l} t - \\ &\mathcal{H}_i x Q_{2,l} t - \varphi x, t + t \psi_{2xx}(x) - x \xi_{2tt} + 1 - x \xi_{1tt} - \alpha x (\varphi_{t1,t} - \psi_{21}) + 1 - x \\ &\varphi_{t0,t} - \psi_{20} + \psi_{2x} - \beta x (\xi_{2t} - \xi_{20}) - x t \psi_{21} + 1 - x \xi_{1t} - \xi_{10} - 1 - x t \psi_{20} + t \psi_{2x} \end{aligned} \quad (4.20)$$

Now discretizing the variable as $x \rightarrow x_r$, $t \rightarrow t_s$ where $x_r = \frac{2r-1}{6p}$, $t_s = \frac{2s-1}{6p}$, $r, s = 1, 2, \dots, 3p$ in the above equations we get the following system of algebraic equations

$$\sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{i\ell} R_{i,l,r,s} = F(r, s) \quad (4.21)$$

Where

$$\begin{aligned} R_{i,l,r,s} &= \left[\left(Q_{2,i}(x_r) - x_r Q_{2,i}(1) \right) \mathcal{H}_\ell(t_s) + \alpha \left(\left(Q_{2,i}(x_r) - x_r Q_{2,i}(1) \right) Q_{1,\ell}(t_s) \right) \right. \\ &\quad \left. + \beta \left(\left(Q_{2,i}(x_r) - x_r Q_{2,i}(1) \right) Q_{2,\ell}(t_s) \right) - \mathcal{H}_i(x_r) Q_{2,\ell}(t_s) \right] \\ F(r, s) &= \varphi(x_r, t_s) + t(\psi_2)_{xx}(x_r) - \left(x_r (\xi_2(t_s))_{tt} + (1 - x_r) (\xi_1(t_s))_{tt} \right) \\ &\quad - \alpha \left(x_r (\varphi_t(1, t_s) - \psi_2(1)) + (1 - x_r) (\varphi_t(0, t_s) - \psi_2(0)) + \psi_2(x_r) \right) \\ &\quad - \beta \left(x_r (\xi_2(t_s) - \xi_2(0)) - x_r t_s \psi_2(1) + (1 - x_r) (\xi_1(t_s) - \xi_1(0)) \right. \\ &\quad \left. - (1 - x_r) t \psi_2(0) + t_s \psi_2(x_r) \right) \end{aligned}$$

The above system reduced to the system of algebraic equations and further it gets reduced to the following system of 4D-arrays

$$A_{3p \times 3p} R_{3p \times 3p \times 3p \times 3p} = F_{3p \times 3p} \quad (4\text{Error!})$$

Further, the above arrays system is reduced to the following matrix system using the transformations

$$\begin{aligned} a_{i\ell} &= b_\lambda \text{ and } F_{rs} = G_\mu \\ B_{1 \times (3p)^2} S_{(3p)^2 \times (3p)^2} &= G_{1 \times (3p)^2} \end{aligned} \quad (4.23)$$

Where $\lambda = 3p(i - 1) + l$ and $\mu = 3p(r - 1) + s$

Then the values of b_λ can be calculated successively for different values of $n=1,2,\dots$ by solving the above system of equation using the Thomas algorithm via MATLAB Program. Original wavelet coefficients $a_{i\ell}$ can be restored using the above transformation. These coefficients will be used in the equations to determine the final solution of the problem) for different value of t_n for $n=0,1, 2,\dots$

5. Error analysis with test problems

Propose scheme is tested on some Telegraph equations to judge the competence of the scheme and level of accuracy obtained by the present scheme. $L_2 - \text{error}$, $L_\infty - \text{error}$ and absolute errors have been calculated for each problem using the present scheme with the help of following formulas

$$\text{Absolute error} = |u_{\text{exact}}(x_r, t_s) - u_{\text{num}}(x_r, t_s)| \quad (5.24)$$

$$L_\infty = \max_{r,s} |u_{\text{exact}}(x_r, t_s) - u_{\text{num}}(x_r, t_s)| \quad (5.25)$$

$$L_2 = \frac{\sqrt{\sum_{l=1}^{3p} |u_{\text{exact}}(x_r, t_s) - u_{\text{num}}(x_r, t_s)|^2}}{\sqrt{\sum_{l=1}^{3p} |u_{\text{exact}}(x_r, t_s)|^2}} \quad (5.26)$$

Test Problem. 1: - Consider the “1D hyperbolic Telegraph equation” of second order

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\alpha \frac{\partial \varphi}{\partial t} + \beta^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + g(x, t), (x, t) \in [0, 1] \times [0, T] \quad (5.27)$$

under the following types of initial constraints

$$\varphi(x, 0) = e^x, \quad x \in [0, 1]$$

$$\frac{\partial \varphi}{\partial t}(x, 0) = -e^x, \quad x \in [0, 1]$$

and the boundary constraints

$$\varphi(0, t) = e^{-t}, \quad t \in [0, T]$$

$$\varphi(1, t) = e^{1-t}, \quad t \in [0, T]$$

with $2\alpha = 1$, $\beta = 1$ and $g(x, t) = 0$ and $\varphi(x, t)$ is a function whose value is to be determined.

Existing solution obtained from the literature for problem 1 is $\varphi(x, t) = e^{x-t}$

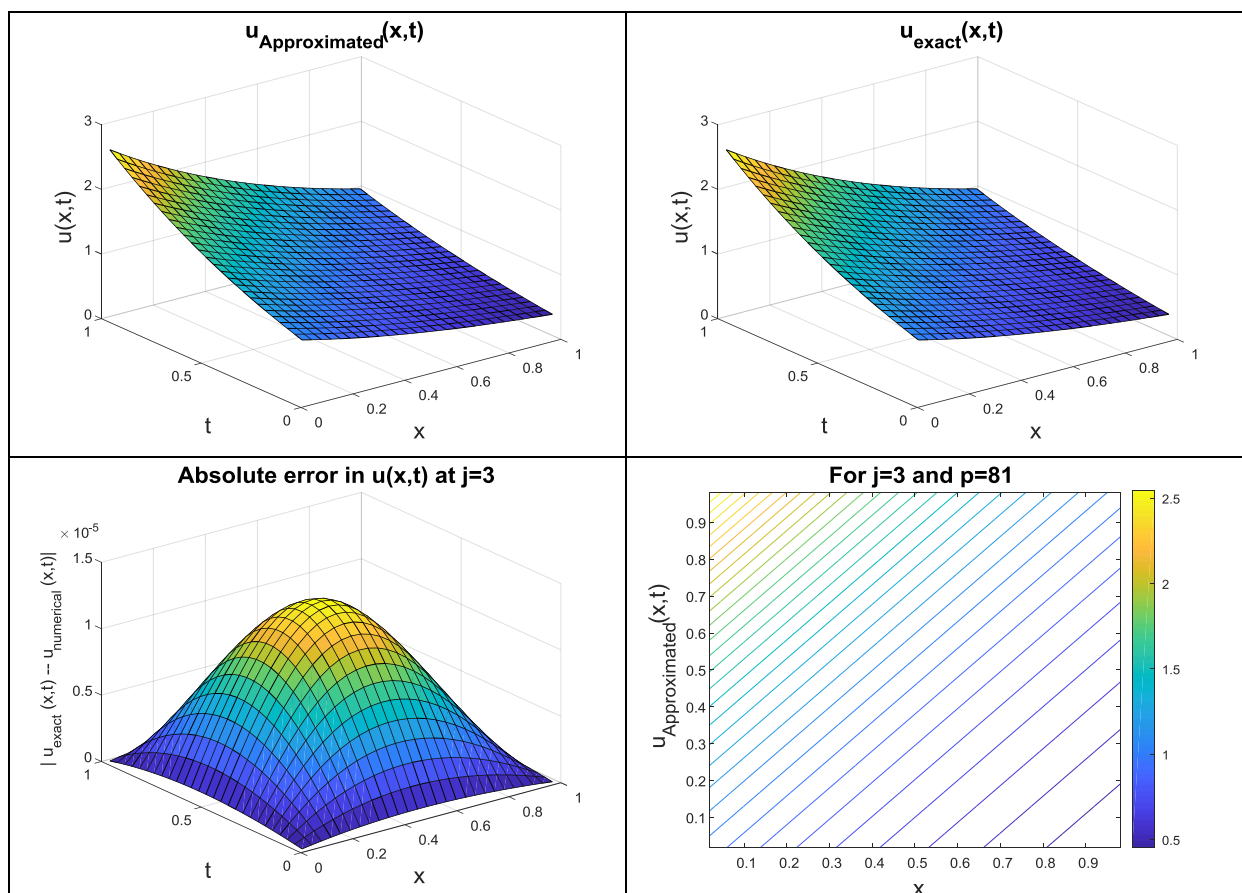


Figure 1: Graphical representation of Approximate solution(fig.1a), Exact solution(fig.1b), Absolute error(fig.1c) and contour view of the approx. solution (fig.1d) in the solution of numerical Experiment no 1.

We proposed the following numerical solution using current scheme

$$\begin{aligned} \varphi(x, t) = & \sum_{i=1}^{3p} \sum_{l=1}^{3p} a_{il} \left(Q_{2,i}(x) - x Q_{2,i}(1) \right) Q_{2,l}(t) + x (\xi_2(t) - \xi_2(0)) - x t \psi_2(1) \\ & + (1-x)(\xi_1(t) - \xi_1(0)) - (1-x) t \psi_2(0) + t \psi_2(x) \end{aligned}$$

a_i 's are the wavelets coefficients which will be obtained by the following procedure and $q_{j,i}$'s are the wavelets integrals which has been already calculated in section 3.

The particular solution is obtained using the boundary condition on the above equation and presented in the form of tables and figures. Figure 1 demonstrate the clearly visible agreement in the exact and approximated solutions. Table 1 is depicting the current method performance in comparison with the exact solution existing the recent literature. We infer that the proposed method is working well. Further It has been observed that the error norms decrease with increase in collocation points which ensures the stability of the proposed numerical scheme. Hence, we can conclude that proposed technique is a strong solver in terms of good accuracy.

Table 1 : Comparison of results achieved for Test problem 1 with Exact solution

x	t	Approximate Solution	Exact solution	Absolute error
0.1	0.1	0.999674	1	3.26E-04
0.2	0.2	0.999229	1	7.71E-04
0.3	0.3	0.999076	1	9.24E-04
0.4	0.4	0.999188	1	8.12E-04
0.5	0.5	0.999310	1	6.90E-04
0.6	0.6	0.999281	1	7.19E-04
0.7	0.7	0.999249	1	7.51E-04
0.8	0.8	0.999416	1	5.84E-04
0.9	0.9	0.999768	1	2.32E-04

Table 2: L_2 and L_∞ errors at different value of j for Test problem 1

Level of resolution	$J = 1$	$J = 2$	$J = 3$
L_2 -error	4.837014364118279e-05	5.426745837985967e-06	6.036516821232161e-07
L_∞ -error	1.151108100422293e-04	1.306821554991622e-05	1.454308205639521e-06

Test Problem. 2: - Consider the “1D hyperbolic Telegraph equation” of second order

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\alpha \frac{\partial \varphi}{\partial t} + \beta^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + g(x, t), (x, t) \in [0, 1] \times [0, T] \quad (5.28)$$

under the following types of initial constraints

$$\varphi(x, 0) = \sin x, \quad x \in [0, 1]$$

$$\frac{\partial \varphi}{\partial t}(x, 0) = -\sin x, \quad x \in [0, 1]$$

and the boundary constraints

$$\varphi(0, t) = 0, \quad t \in [0, T]$$

$$\varphi(1, t) = e^{-t} \sin 1, \quad t \in [0, T]$$

with $\alpha = 2$, $\beta^2 = 2$ and $g(x, t) = 0$ and $\varphi(x, t)$ is the unknown function whose value is to be determined.

Existing solution obtained from the literature for problem 2 is $\varphi(x, t) = e^{-t} \sin x$

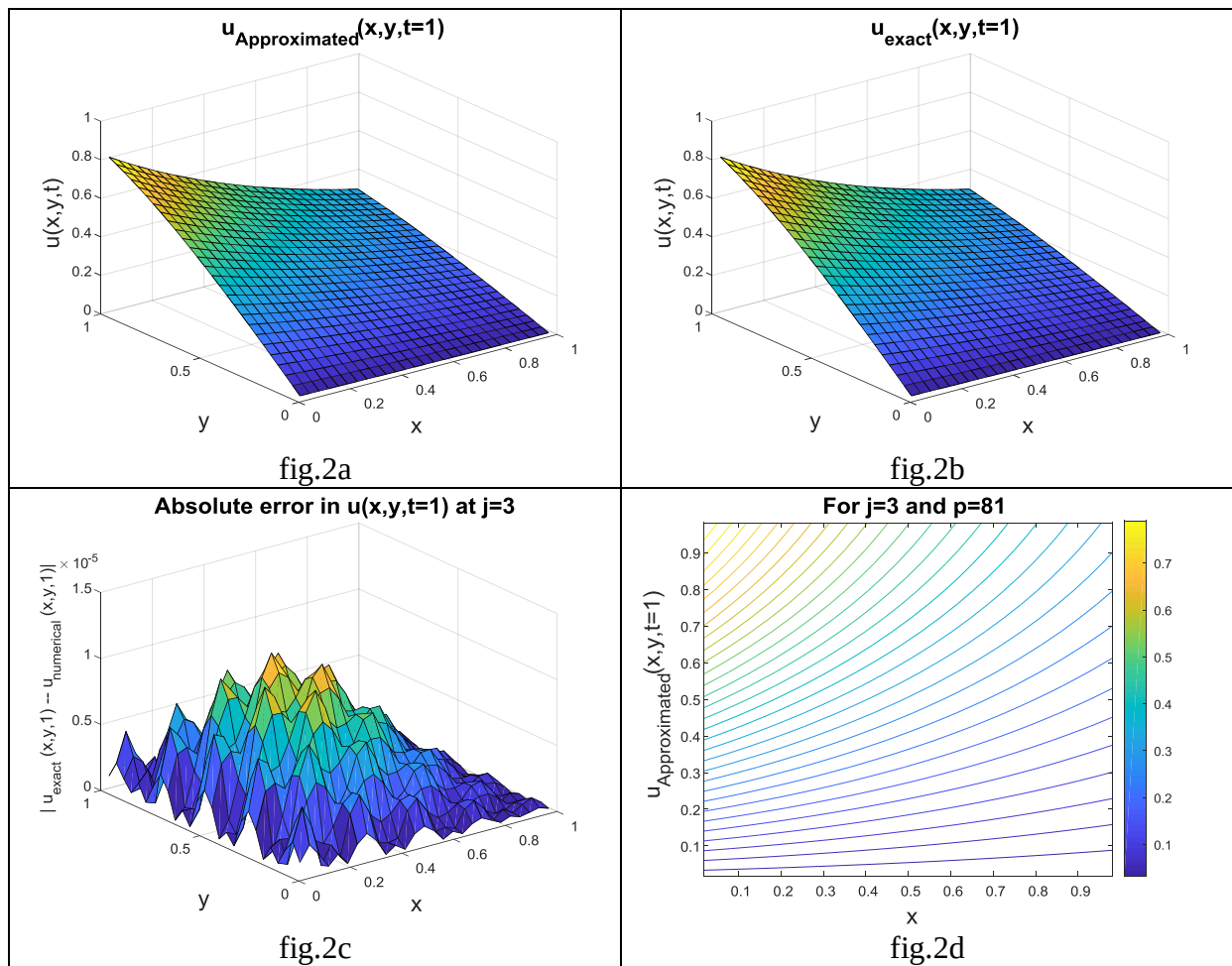


Figure 2: Graphical representation of Approximate solution(fig.2a), Exact solution(fig.2b), Absolute error(fig.2c) and contour view of the approx. solution (fig.2d) in the solution of numerical Experiment no 2.

Table 3: Comparison of results achieved for Test problem 2 with Exact solution

x	t	Approximate Solution	Exact solution	Absolute error
0.1	0.1	0.089473	0.090333	8.60E-04
0.2	0.2	0.161207	0.162657	1.45E-03
0.3	0.3	0.217964	0.218927	9.63E-04
0.4	0.4	0.260608	0.261035	4.27E-04
0.5	0.5	0.290540	0.290786	2.47E-04
0.6	0.6	0.309681	0.309882	2.02E-04
0.7	0.7	0.319588	0.319909	3.21E-04
0.8	0.8	0.321902	0.322329	4.27E-04
0.9	0.9	0.318267	0.318477	2.10E-04

Table 4: L_2 and L_∞ errors at different value of j for Test problem 2

Level of resolution	$J = 1$	$J = 2$	$J = 3$
L_2 -error	2.966139636402692e-04	1.412472863080943e-05	1.508868064851954e-06
L_∞ -error	3.086337934932737e-04	1.238748966753134e-05	1.129461926263620e-06

Results obtained by the proposed scheme has been explained with the help of tables and surface plots. It is very much clear from Figure 2 that results achieved with the proposed scheme are roughly coinciding with the analytic solution. The level of accuracy obtained for the solution is of order 10^{-4} which is a noteworthy achievement for these kinds of problems. The solution achieved can further be improved by increasing the number of collocation points. In table 3 error norms obtained by the present scheme are presented in comparison with the exact solution. It has been observed that with the increase in the collocation points (level of resolution), error norms are decreasing which ensure the stability of the proposed scheme. We infer that our scheme is working well.

Test Problem 3: - Consider the “1D hyperbolic Telegraph equation” of second order

$$\frac{\partial^2 \varphi}{\partial t^2} + 2\alpha \frac{\partial \varphi}{\partial t} + \beta^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + g(x, t), (x, t) \in [0, 1] \times [0, T] \quad (5.29)$$

under the following types of initial constraints

$$\varphi(x, 0) = x^2, \quad x \in [0, 1]$$

$$\frac{\partial \varphi}{\partial t}(x, 0) = 1, \quad x \in [0, 1]$$

and the boundary constraints

$$\varphi(0, t) = t, \quad t \in [0, T]$$

$$\varphi(1, t) = 1 + t, \quad t \in [0, T]$$

with $\alpha = 1$, $\beta = 1$ and $g(x, t) = x^2 + t - 1$ and $\varphi(x, t)$ is the unknown function whose value is to be determined.

Existing solution obtained from the literature for problem 3 is $\varphi(x, t) = x^2 + t$.

It is shown in Table 5 and Figure 3 that results achieved with the proposed technique exactly matching with exact solution with no error. Fig 9 explains the high level of agreement between the exact and approximated solution. Getting high accuracy for a small number of grid points makes it strong solver for these kinds of mathematical models.

Table 5: Comparison of results achieved for Test problem 3 with Exact solution

x	t	Approximate Solution	Exact solution	Absolute error
0.1	0.1	0.3	0.3	0
0.2	0.2	0.6	0.6	0
0.3	0.3	0.9	0.9	0
0.4	0.4	1.2	1.2	0
0.5	0.5	1.5	1.5	0
0.6	0.6	1.8	1.8	0
0.7	0.7	2.1	2.1	0

0.8	0.8	2.4	2.4	0
0.9	0.9	2.7	2.7	0

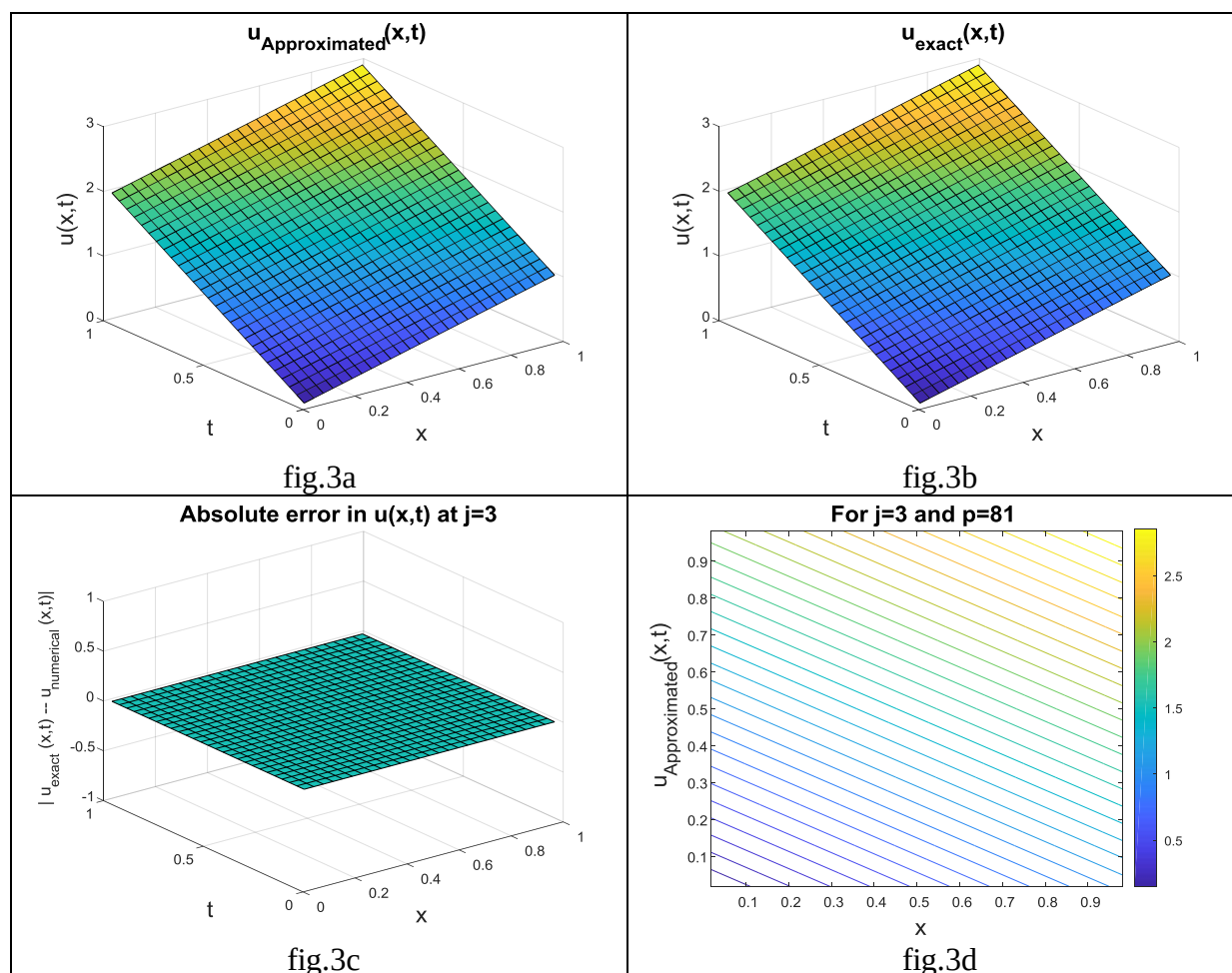


Figure 3: Graphical representation of Approximate solution(fig.3a), Exact solution(fig.3b), Absolute error(fig.3c) and contour view of the approx. solution (fig.3d) in the solution of numerical Experiment no 3.

6. Conclusion

After looking at the results of numerical experiments performed on three test problems with proposed technique, we infer that 1D- hyperbolic Telegraph equations of second order can easily be solved by the proposed scheme with less computational cost and high accuracy. level of accuracy obtained is or order 10^{-4} for only 9 collocation points in the first iteration is a noteworthy achievement for these kinds of equations. Moreover, the use of common MATLAB subprograms to solve various types of Telegraph equations, makes it more computer friendly. Very good accuracy is obtained for a very small number of collocation points makes the proposed scheme a strong solver for these kinds of differential equations. Therefore, by looking at the performance of the method, we conclude that the given method can be extended to solve other set of differential equations. All the calculations have been performed using the MATLAB 7.

7. References

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