

Analysis of Love Type Waves In An Isotropic Elastic Layer Lying Over Couple Stress Substrate

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Abstract: This paper deals with the study of Love type wave propagation in a finite thicker homogeneous isotropic layer resting over a microstructural substrate which is mathematically governed by consistent couple stress theory. Mathematical formulations of consistent couple stress theory involve a characteristic length parameter and it is of order of cell size of material. Characteristic length parameter predicts size dependent response of the material. Dispersion equation for the propagation Love waves is derived by applying suitable boundary conditions. Impacts of thickness of isotropic layer and characteristic length parameter of substrate are studied on the phase velocity of Love waves.

Keywords: Love waves, Consistent couple stress theory, Characteristic length, Isotropic elastic material

1. Introduction

Love waves are transverse horizontal waves, which propagate through a layered media having a finite thicker layer attached to a half-space. Propagation of Love waves requires a velocity push when they propagate into the depth of medium. Study of these waves is important in seismology, earthquake engineering and non-destructive evaluation techniques. Love type waves are studied by many researchers in different geometries involving various materials such as Sharma and Kumar [8], Goyal *et al.* [5], Singh *et al.* [1].

Microcontinuum theories are generalizations of classical mechanics, involving some additional material parameters as compared to classical theory. These additional parameters in the

formulations of microcontinuum theories estimate size dependent response of the materials. Consistent couple stress theory [2] is a microcontinuum theory which involves an additional material parameter called couple stress coefficient (η_c) and it further is dependent upon a material length parameter called characteristic length of the material (l), which is of order of average internal cell size of material. This parameter creates a difference between classical theory of elasticity and consistent couple stress theory. Moreover, stresses defined in constitutive relations of consistent couple stress theory are also non-symmetric in nature. Constitutive relations involve an additional expression defining couple stresses in terms of displacement vector and these couple stresses are skew-symmetric in nature. Researchers have used consistent couple stress theory to study various problems such as Sharma and Kumar [6-7], Fan and Xu [3].

2. Formulation and solution of the problem

Love waves are propagating in an isotropic homogeneous elastic layer of finite width which is bonded to a half space with microstructures and half space is mathematically governed by consistent couple stress theory. We are considering that origin of coordinate system lies at the interface between layer and half space, we further assume that x –axis is positive vertically downwards into the half space, z –axis is pointing along the propagation of wave. The interface between layer and half space is defined as $x = 0$ and the upper layer is assumed to be of finite width, so it is defined as $x = -H$. For the propagation of Love waves in the considered geometry, we assume that displacement vectors are taken as $\vec{u} = (u_c, v_c, w_c) = (0, v_c, 0)$ for the half space and $\vec{u}_e = (u_1, v_1, w_1) = (0, v_1, 0)$ for isotropic elastic layer and we further assume that $\frac{\partial}{\partial y} \equiv 0$.

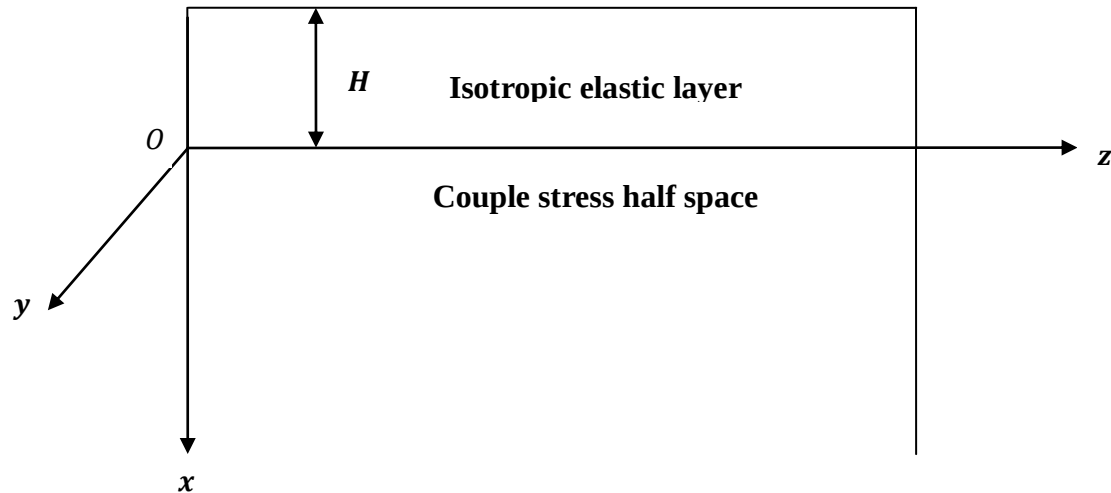


Figure 1: Geometry of the Problem

2.1 Solution of consistent couple stress half space

The governing equation for consistent couple stress half space is given as [2]

$$(\lambda_c + \mu_c + \eta_c \nabla^2) \nabla(\nabla \cdot \vec{u}) + (\mu_c - \eta_c \nabla^2) (\nabla^2 \vec{u}) = \rho_c \frac{\partial^2 \vec{u}}{\partial t^2} \quad (1)$$

Imposing the conditions $\vec{u} = (u_c, v_c, w_c) = (0, v_c, 0)$ and $\frac{\partial}{\partial y} \equiv 0$, the equation (1) becomes,

$$\left(\frac{\partial^2 v_c}{\partial x^2} + \frac{\partial^2 v_c}{\partial z^2} \right) - l^2 \left(\frac{\partial^4 v_c}{\partial x^4} + 2 \frac{\partial^4 v_c}{\partial x^2 \partial z^2} + \frac{\partial^4 v_c}{\partial z^4} \right) = \frac{\rho_c}{\mu_c} \frac{\partial^2 v_c}{\partial t^2} \quad (2)$$

Let $v_c = \phi(x) e^{ik(z-ct)}$ be the solution, so equation (2) becomes

$$\frac{d^4 \phi}{dx^4} - \beta_2 \frac{d^2 \phi}{dx^2} + \beta_3 \phi(x) = 0 \quad (3)$$

where, $\beta_2 = \left(2k^2 + \frac{1}{l^2} \right)$ and $\beta_3 = \left(k^4 + \frac{k^2}{l^2} - \frac{k^2 c^2}{l^2 \beta_c^2} \right)$, $\beta_c = \sqrt{\frac{\mu_c}{\rho_c}}$

Consider the auxiliary equation for differential equation (3),

$$D^4 - \beta_2 D^2 + \beta_3 = 0 \quad \text{where } D \equiv \frac{d}{dx}$$

The solution of equation (3) is, $\phi(x) = B_4 e^{b_1 x} + B_5 e^{-b_1 x} + B_6 e^{b_2 x} + B_7 e^{-b_2 x}$

where, $b_1 = \sqrt{\frac{B_2 + \sqrt{B_2^2 - 4B_3}}{2}}$, $b_2 = \sqrt{\frac{B_2 - \sqrt{B_2^2 - 4B_3}}{2}}$ and B_4, B_5, B_6 and B_7 are arbitrary constants.

Hence, $v_c = \phi(x)e^{ik(z-ct)} = (B_4e^{b_1x} + B_5e^{-b_1x} + B_6e^{b_2x} + B_7e^{-b_2x})e^{ik(z-ct)}$

Since with the depth in half space wave gradually dies down, so to meet the condition that $v_c \rightarrow 0$, as $x \rightarrow \infty$, we take $B_4 = B_6 = 0$, therefore we get,

$$v_c = (B_5e^{-b_1x} + B_7e^{-b_2x})e^{ik(z-ct)} \quad (4)$$

The constitutive relations for consistent couple stress substrate are given by [2]

$$\sigma_{ji} = \lambda_c u_{k,k} \delta_{ij} + \mu_c (u_{i,j} - u_{j,i}) - \eta_c \nabla^2 (u_{i,j} - u_{j,i}) \quad (5)$$

$$\mu_{ji} = 4\eta_c (\omega_{i,j} - \omega_{j,i}) \quad (6)$$

where, $\omega_i = \frac{1}{2} \epsilon_{ijk} u_{k,j}$ where ϵ_{ijk} is permutation tensor and $i, j, k = 1, 2, 3$.

By taking the $\vec{u} = (u_c, v_c, w_c) = (0, v_c, 0)$ and $\frac{\partial}{\partial y} \equiv 0$, we get

$$\sigma_{xy} = \mu_c \left(\frac{\partial v_c}{\partial x} \right) - \eta_c \nabla^2 \left(\frac{\partial v_c}{\partial x} \right)$$

$$\sigma_{xy} = [(-\mu_c b_1 + \eta_c b_1^3 - \eta_c k^2 b_1)B_5 e^{-b_1x} + (-\mu_c b_2 + \eta_c b_2^3 - \eta_c k^2 b_2)B_7 e^{-b_2x}]e^{ik(z-ct)} \quad (7)$$

$$\mu_{xz} = 2\eta_c \left[\frac{\partial^2 v_c}{\partial x^2} + \frac{\partial^2 v_c}{\partial z^2} \right]$$

$$\mu_{xz} = 2\eta_c \{ [B_5 b_1^2 e^{-b_1x} + B_7 b_2^2 e^{-b_2x}]e^{ik(z-ct)} - (B_5 e^{-b_1x} + B_7 e^{-b_2x})k^2 e^{ik(z-ct)} \} \quad (8)$$

2.2 Solution of homogeneous isotropic elastic layer

The governing equation for homogeneous isotropic elastic layer is given by

$$(\lambda_1 + \mu_1) \text{grad div } \vec{u}_e + \mu_1 \nabla^2 \vec{u}_e = \rho_1 \frac{\partial^2 \vec{u}_e}{\partial t^2} \quad (9)$$

λ_1, μ_1 are lame parameters and $\vec{u}_e = (u_1, v_1, w_1) = (0, v_1, 0)$ is the displacement vector and $\frac{\partial}{\partial y} \equiv 0$, so equation (9) becomes

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{\alpha_1^2} \frac{\partial^2 v_1}{\partial t^2}, \text{ where } \alpha_1 = \sqrt{\frac{\mu_1}{\rho_1}} \quad (10)$$

Now, let us assume that solution of equation (10) as $v_1 = \psi_1(x)e^{ik(z-ct)}$, we get

$$\frac{d^2 \psi_1}{dx^2} + \left(\frac{k^2 c^2}{\alpha_1^2} - k^2 \right) \psi_1 = 0$$

$$\frac{d^2 \psi_1}{dx^2} + a_1^2 \psi_1 = 0, \text{ where } a_1^2 = k^2 \left(\frac{c^2}{\alpha_1^2} - 1 \right)$$

Auxiliary equation of above differential equation is given as $(D^2 + a_1^2) = 0$ where $D \equiv \frac{d}{dx}$, hence roots of auxiliary equation are $D = \pm ia_1$, so solution of above differential equation becomes

$$\psi_1(x) = A_2 \cos(a_1 x) + A_3 \sin(a_1 x)$$

$$\text{Hence, } v_1 = [A_2 \cos(a_1 x) + A_3 \sin(a_1 x)] e^{ik(z-ct)} \quad (11)$$

Here, constitutive relation for isotropic elastic layer is given by

$$S_{ij} = 2\mu_1 e_{ij} + \lambda_1 \vartheta \delta_{ij} \text{ where } e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \vartheta = e_{kk} = \frac{\partial u_k}{\partial x_k} \quad (12)$$

By imposing the conditions $\vec{u}_e = (u_1, v_1, w_1) = (0, v_1, 0)$ and $\frac{\partial}{\partial y} \equiv 0$, we get

$$S_{xy} = \mu_1 \left(\frac{\partial v_1}{\partial x} \right) = \mu_1 a_1 [-A_2 \sin(a_1 x) + A_3 \cos(a_1 x)] e^{ik(z-ct)} \quad (13)$$

3. Boundary conditions

Boundary conditions to be applied are

- i. $S_{xy} = 0$ at $x = -H$
- ii. $S_{xy} = \sigma_{xy}$ at $x = 0$
- iii. $v_1 = v_c$ at $x = 0$

iv. $\mu_{13} = 0$ at $x = 0$

By applying above mentioned boundary conditions, we obtain following four homogeneous linear equations

$$A_2 \sin(a_1 H) + A_3 \cos(a_1 H) = 0 \quad (14)$$

$$\mu_1 a_1 A_3 + (\mu_c b_1 - \eta_c b_1^3 + \eta_c k^2 b_1) B_5 + (\mu_c b_2 - \eta_c b_2^3 + \eta_c k^2 b_2) B_7 = 0 \quad (15)$$

$$A_2 - B_5 - B_7 = 0 \quad (16)$$

$$B_5(b_1^2 - k^2) + B_7(b_2^2 - k^2) = 0 \quad (17)$$

For non-trivial solution of above system of four homogeneous equations, the determinant of coefficient matrix of four unknowns A_2, A_3, B_5, B_7 must vanish, so by imposing stated condition, we get following dispersion equation for the propagation of Love waves in the considered geometry as

$$\sin(a_1 H) \mu_1 a_1 (b_1^2 - b_2^2) + \cos(a_1 H) \left[\frac{(b_1 \mu_c - \eta_c b_1^3 + \eta_c b_1 k^2)(b_2^2 - k^2) - (b_2 \mu_c - \eta_c b_2^3 + \eta_c b_2 k^2)(b_1^2 - k^2)}{(b_2 \mu_c - \eta_c b_2^3 + \eta_c b_2 k^2)(b_1^2 - k^2)} \right] = 0 \quad (18)$$

4. Numerical results and discussion

For graphical illustrations of the results we have taken following data for material parameters

(i). For isotropic elastic layer [4], $\mu_1 = 3.91 \times 10^{10} \text{ N/m}^2$, $\rho_1 = 8900 \text{ kg/m}^3$

(ii). For couple stress substrate [8], $\mu_c = 3.05 \times 10^{10} \text{ N/m}^2$, $\rho_c = 2717 \text{ kg/m}^3$

It can be seen that phase velocity of Love waves is decreasing with the increase in the value of wave number, hence Love waves are dispersive for the considered geometry. Figure 1 shows the impacts of thickness of isotropic layer on the phase velocity of Love waves in the considered geometry. Figure 1 is plotted between normalized phase velocity (c/α_1) and normalized wave number (kl). Effects are shown by taking four different values of thickness of isotropic layer as $H = 0.0002 \text{ m}, 0.0003 \text{ m}, 0.0004 \text{ m}, 0.0005 \text{ m}$, it is found that phase velocity decreases with the increase in thickness of isotropic layer. Hence thickness of isotropic layer goes against the phase velocity. Figure 2 shows the impact of microstructural parameter called characteristic length (l) of the substrate on phase velocity of Love waves, again graph is plotted between dimensionless phase velocity (c/α_1) and normalized wave number (kH). It is observed that with increase ($l = 0.00001 \text{ m}, 0.00005 \text{ m}, 0.00009 \text{ m}$) in value of characteristic length parameter, phase velocity is also increasing. Hence characteristic length parameter favors phase velocity.

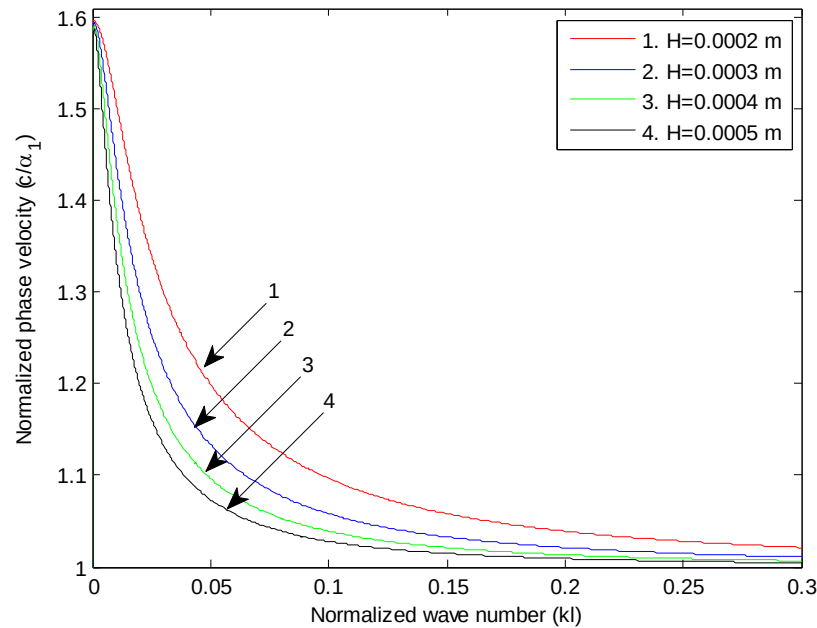


Figure 2: Normalized phase velocity (c/α_1) versus normalized wave number (kl) for different values of thickness of layer (H)

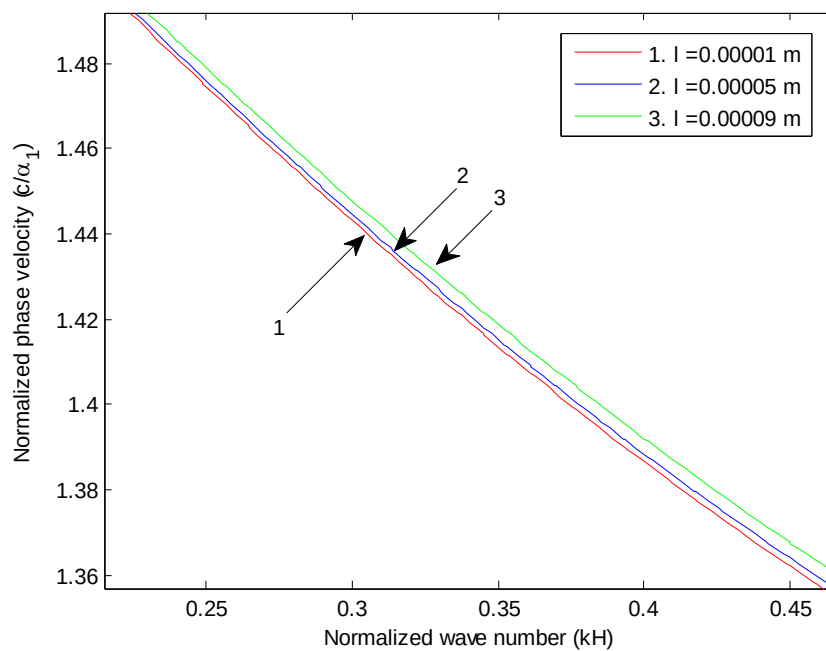


Figure 3: Normalized phase velocity (c/α_1) versus normalized wave number (kl) for different values of characteristic length (l)

5. Conclusion: The paper deals with the study about Love type waves in a homogeneous isotropic layer attached perfectly to an underlying microstructural substrate. The substrate is modeled using consistent couple stress theory, which involves an additional parameter known as characteristic length. From the analysis, it is found that Love waves are dispersive in the considered geometry. It is observed that thickness of isotropic layer goes against phase velocity of Love waves and phase velocity decreases with the increase in thickness of layer. It is further found that characteristic length parameter of substrate favors phase velocity of Love waves. Present analysis may find certain applications in seismology, non-destructive testing techniques which use Love type waves to find surface flaws in the materials.

6. References

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