ISSN: 0971-1260
Vol-22-Issue-17-September-2019

Generalization of A Known Congruence By Andrews, Dixit and Yee Related To The Mock Theta Function $\nu(q)$

MANDEEP KAUR

Department of Mathematics, School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara 144411, Punjab, India.

Email: mandeep.25178@lpu.co.in

Abstract. We find the exact generating function for $p_\nu(10n+8)$ from where the congruence $p_\nu(10n+8) \equiv 0(mod\ 5)$ follows immediately.

Key Words: partitions, mock theta functions.

2010 Mathematical Reviews Classification Numbers: Primary 11P83; Secondary 05A15, 05A17.

1. INTRODUCTION

A q series has an expression involving $(a;q)_n$, which is defined by

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), n \geq 1$$

and$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), |q| < 1$,

where$a$ is any complex number.

Ramanujan'sgeneral theta function $f(a,b)$ is defined by

$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2}\, b^{k(k-1)/2}$, $|ab| < 1$.

In this notation, Jacobi's famous triple product identity takes the form

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \qquad (1.1)$$

Three special cases of $f(a,b)$ are [6, p. 36, Entry 22]

$$\varphi(q) := f(q,q) = \sum_{k=-\infty}^{\infty} q^{k^2} = (-q;q^2)_\infty^2(q^2,q^2)_\infty,$$

$$\psi(q) := f(q,q^3) = \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \qquad (1.2)$$

and

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2} = (q;q)_\infty,$$

where the product representations in the above arise from (1.1). After Ramanujan, we also define

$$\chi(q) := (-q;q^2)_\infty. \qquad (1.3)$$

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Vol-22-Issue-17-September-2019

A partition $\lambda = (\lambda_1, \lambda_2, \dots \dots, \lambda_k)$ of a non-negative integer n is a non-increasing sequence of positive integer parts such that $n = \sum_{i=0}^{k} \lambda_i$. The number of partitions of n is called the partition function and is denoted by $p(n)$. By convention, $p(0) = 1$. For example, $p(4) = 5$, since there are 5 partitions of 4, namely, 4, 3 + 1, 2 + 2, 2 + 1 + 1, 1 + 1 + 1 + 1.

The generating function for $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}.$$

In his last letter to Hardy, Ramanujan defined 17 theta function like functions, for $|q| < 1$, which he called mock theta functions. Ramanujan found an additional three mock theta functions in his lost notebook [1]. See [2] and [3] for details on the subject.

In [4], Andrews, Dixit and Yee found many results related to the mock theta functions $\omega(q)$, $\nu(q)$ and $\phi(q)$. These three mock theta functions are defined, by Ramanujan, as

$$\omega(q) = \sum_{n=0}^{\infty} \frac{q^{2(n^2+n)}}{(q;q^2)_{n+1}^2},$$

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(-q;q^2)_{n+1}},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2;q^2)_n}.$$

Andrew et al. [4] proved that

$$\sum_{n=0}^{\infty} p_\omega(n)q^n = q\omega(q),$$

$$\sum_{n=0}^{\infty} p_\nu(n)q^n = \nu(-q),$$

where, $p_\omega(n)$ denotes the number of partitions of n in which each odd part is less than twice the smallest part and $p_\nu(n)$ denotes the number of partitions of n into distinct parts in which each odd part is less than twice the smallest part.

The simpler form of these two functions as given in [4] are

$$\sum_{n=1}^{\infty} p_\omega(n)q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)(q^{n+1};q)_n(q^{2n+2};q^2)_\infty},$$

$$\sum_{n=0}^{\infty} p_\nu(n)q^n = q\omega(q^2) + (-q^2;q^2)_\infty \psi(q^2).$$

Andrews, Dixit and Yee [4] find that

$$p_\nu(10n+8) \equiv 0 (mod\ 5).$$

Theorem 1.1.

$$\sum_{n=0}^{\infty} p_v(n)q^n = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}, \qquad (1.4)$$

where $f_k := (q^k; q^k)_\infty$.

2. Preliminaries

Lemma 2.1.

$$xy^2 - \frac{q^2}{xy^2} = k, \qquad (2.1)$$

$$\frac{x^2}{y} - \frac{y}{x^2} = 4\frac{q}{k}, \qquad (2.2)$$

$$x^5 - \frac{q^2}{x^5} = 11q + \frac{f_1^6}{f_5^6}, \qquad (2.3)$$

$$y^5 - \frac{q^4}{y^5} = 11q^2 + \frac{f_2^6}{f_{10}^6}. \qquad (2.4)$$

where x and y are defined as-

$x = T(q^5)$,

$y = T(q^{10})$.

Here $T(q)$ is a continued fraction given by

$$T(q) := 1 + \frac{q}{1+\frac{q^2}{1+\frac{q^3}{1+\cdots}}}.$$

Lemma 2.2.

$$\frac{f_5^7}{f_1^4 f_{10}^3} - 4q\frac{f_5^2 f_{10}^2}{f_1^3 f_2} = \frac{f_5^3}{f_2^2 f_{10}}, \qquad (2.5)$$

$$\frac{f_2^4}{f_1^2} - 5q\frac{f_{10}^4}{f_5^2} = \frac{f_1^3 f_{10}}{f_2 f_5}, \qquad (2.6)$$

$$\frac{f_5^7}{f_1^4 f_{10}^3} + q\frac{f_5^2 f_{10}^2}{f_1^3 f_2} = \frac{f_2^3 f_5^4}{f_1^5 f_{10}^2}, \qquad (2.7)$$

$$\frac{f_1^4}{f_2^2} + 4\frac{f_2^3 f_5}{f_1 f_{10}} = \frac{f_5^4}{f_{10}^2}. \qquad (2.8)$$

3. Proof of Theorem1.1

We have,

$$\sum_{n=0}^{\infty} p_v(n)q^n = q\omega(q^2) + \psi(q^2)(-q^2; q^2)_\infty ,$$

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Which implies

$$\sum_{n=0}^{\infty} p_\nu(2n)\, q^n = \psi(q)(-q;q)_\infty = \frac{f_2^3}{f_1^2}, \tag{3.1}$$

Now, from [5]

$$\frac{1}{f_1} = \frac{f_{25}^5}{f_5^6}\left(x^4 + qx^3 + 2q^2x^2 + 3q^3x + 5q^4 - 3\frac{q^5}{x} + 2\frac{q^6}{x^2} - \frac{q^7}{x^3} + \frac{q^8}{x^4}\right), \tag{3.2}$$

$$f_1 = f_{25}\left(x - q - \frac{q^2}{x}\right), \tag{3.3}$$

and

$$f_2 = f_{50}\left(y - q^2 - \frac{q^4}{y}\right). \tag{3.4}$$

Using (3.2) and (3.4) in (3.1), and extracting the coefficients of q^{5n+4}, we have

$$\sum_{n=0}^{\infty} p_\nu(10n+8)q^n = 5\frac{f_{10}^3 f_5^{10}}{f_1^1 2}[4\left(x^4y^3 - \frac{q^4}{x^4y^3}\right) - 4q\left(\frac{y^3}{x} + q^2\frac{x}{y^3}\right) + q^2\left(\frac{y^3}{x^6} - \frac{x^6}{y^3}\right) - 3\left(x^6y^2 + \frac{q^4}{x^6y^2}\right) - 12q\left(xy^2 - \frac{q^2}{xy^2}\right) + 10q\left(x^5 - \frac{q^2}{x^5}\right) + 15q^2] \tag{3.5}$$

Again using (2.1), (2.2), (2.3) and (2.4) in equation (3.5), we arrive at

$$\sum_{n=0}^{\infty} p_\nu(10n+8)q^n = 5[147q^2\frac{f_{10}^3 f_5^{10}}{f_1^{12}} - 33q\frac{f_2 f_5^{15}}{f_1^{13} f_{10}^2} + 4q^3\frac{f_5^5 f_{10}^8}{f_1^{11} f_2} - 64q^5\frac{f_{10}^1 8}{f_1^9 f_2^3 f_5^5} + 4\frac{f_2^6 f_5^{10}}{f_1^{12} f_{10}^3} - 3\frac{f_2 f_5^9}{f_1^7 f_{10}^2} + 10q\frac{f_5^4 f_{10}^3}{f_1^6} - 240q^4\frac{f_{10}^{13}}{f_1^{10} f_2^2}] \tag{3.6}$$

Now, we use (2.5), (2.6), (2.7) and (2.8), and will try to reduce the powers of f_1 and f_2. Then equation (3.6) becomes

$$\sum_{n=0}^{\infty} p_\nu(10n+8)q^n = 5[-33q\frac{f_5^{11}}{f_1^9 f_2} + 15q^2\frac{f_5^6 f_{10}^5}{f_1^8 f_2^2} + 64q^3\frac{f_5 f_{10}^{10}}{f_1^7 f_2^3} + 16q^4\frac{f_{10}^{15}}{f_1^6 f_2^4 f_5^4} + 10q\frac{f_5^4 f_{10}^3}{f_1^6} + 4\frac{f_5^{10} f_2^6}{f_1^{12} f_{10}^3} - 3\frac{f_5^9 f_2}{f_1^7 f_{10}^2}]$$

 ISSN: 0971-1260
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$$= 5[-33q\frac{f_5^{11}}{f_1^9 f_2} + 15q^2\frac{f_5^6 f_{10}^5}{f_1^8 f_2^2} + 64q^3\frac{f_5 f_{10}^{10}}{f_1^7 f_2^3} + 16q^4\frac{f_{10}^{15}}{f_1^6 f_2^4 f_5^4} + 10q\frac{f_5^4 f_{10}^3}{f_1^6}$$

$$+5\frac{f_5^{13} f_2^3}{f_1^{11} f_{10}^4} - 4\frac{f_2 f_5^9}{f_1^7 f_{10}^2}]$$

$$= 5[-33q\frac{f_5^{11}}{f_1^9 f_2} + 15q^2\frac{f_5^6 f_{10}^5}{f_1^8 f_2^2} + 64q^3\frac{f_5 f_{10}^{10}}{f_1^7 f_2^3} + 16q^4\frac{f_{10}^{15}}{f_1^6 f_2^4 f_5^4} + 10q\frac{f_5^4 f_{10}^3}{f_1^6}$$

$$+5\frac{f_5^9 f_2}{f_1^7 f_{10}^2} + 20q\frac{f_2^2 f_5^8 f_{10}}{f_1^{10}}]$$

$$= 5[-8q\frac{f_5^{11}}{f_1^9 f_2} + 15q^2\frac{f_5^6 f_{10}^5}{f_1^8 f_2^2} + 64q^3\frac{f_5 f_{10}^{10}}{f_1^7 f_2^3} + 16q^4\frac{f_{10}^{15}}{f_1^6 f_2^4 f_5^4} + 10q\frac{f_5^4 f_{10}^3}{f_1^6}$$

$$+\frac{f_5^9 f_2}{f_1^7 f_{10}^2} - 5q\frac{f_5^7 f_{10}^2}{f_1^5 f_2^3}]$$

$$= 5[-13q\frac{f_5^7 f_{10}^2}{f_1^5 f_2^3} - 17q^2\frac{f_5^2 f_{10}^7}{f_1^4 f_2^4} - 4q^3\frac{f_{10}^{12}}{f_1^3 f_2^5 f_5^3} + 10q\frac{f_5^4 f_{10}^3}{f_1^6} + \frac{f_2 f_5^9}{f_1^7 f_{10}^2}]$$

$$= 5[-8q\frac{f_5^7 f_{10}^2}{f_1^5 f_2^3} - 17q^2\frac{f_5^2 f_{10}^7}{f_1^4 f_2^4} - 4q^3\frac{f_{10}^{12}}{f_1^3 f_2^5 f_5^3} + 10q\frac{f_5^4 f_{10}^3}{f_1^6} + \frac{f_5^8}{f_1^2 f_2^4 f_{10}}]$$

$$= 5[-8q\frac{f_5^7 f_{10}^2}{f_1^5 f_2^3} + 33q^2\frac{f_5^2 f_{10}^7}{f_1^4 f_2^4} - 4q^3\frac{f_{10}^1 2}{f_1^3 f_2^5 f_5^3} + \frac{f_5^8}{f_1^2 f_2^4 f_{10}} + 10q\frac{f_5^3 f_{10}^4}{f_1 f_2^5}]$$

$$= 5[-8q\frac{f_5^3 f_{10}^4}{f_1 f_2^5} + q^2\frac{f_{10}^9}{f_2^6 f_5^2} + \frac{f_5^8}{f_1^2 f_2^4 f_{10}} + 10q\frac{f_5^3 f_{10}^4}{f_1 f_2^5}]$$

$$= 5[2q\frac{f_5^3 f_{10}^4}{f_1 f_2^5} + \frac{f_5^8}{f_1^2 f_2^4 f_{10}} + q^2\frac{f_{10}^9}{f_2^6 f_5^2}]$$

$$= \frac{5}{f_{10}}\left[\frac{f_5^4}{f_1 f_2^2} + q\frac{f_{10}^5}{f_2^3 f_5}\right]^2. \qquad (3.7)$$

Now, we have the relation from [6], we have

$$5\phi^2(-q^5) - \phi^2(-q) = 4\chi(-q)\chi(-q^5)\psi^2(q)$$

which implies

$$\frac{f_5^4}{f_1 f_2^2} + q\frac{f_{10}^5}{f_2^3 f_5} = \frac{f_2 f_5 f_{10}}{f_1^2} \qquad (3.8)$$

Now, finally using (3.8) in (3.7), we arrive at

$$\sum_{n=0}^{\infty} p_v(10n+8)q^n = 5\frac{f_2^2 f_5^2 f_{10}}{f_1^4}.$$

4. Conclusion

ISSN: 0971-1260
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We used Ramanujan's Continued fractions and five dissection of f_1 and $\frac{1}{f_1}$ to find the exact generating function of $p_\nu(10n+8)$.

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