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# Radial Basis Function collocation method for the numerical solution of Newell-Whitehead-Segal equation

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## *Abstract*

*In this manuscript, a hybrid technique based on radial basis function is used for the numerical solution of Newell-Whitehead-Segal (NWS) equation. Radial basis function gives its best when combined with pseudospectral approach. Using radial basis function, the given NWS equation is transformed to a system of ODEs which can be easily solved with the help of an ODE solver. Numerical examples are established to showcase the validness and efficiency of the method. Error norms are calculated to check the accuracy. The obtained numerical results are in good accord with the already published results in literature.*

***Keywords:*** *Radial basis function**,** Collocation,Pseudospectral approach*

## 1. Introduction

Various Phenomena arising in engineering and applied sciences are modelled with the support of non-linear partial differential equations (PDEs). In mathematical analysis, solving the non-linear partial differential equation is an important task. Many real life problems in every field of science are modelled through PDEs. As most of the non-linear PDEs do not have an analytic solution so numerical methods are applied to such equations to find the solution. Numerical methods can be treated as an important tool in solving complexPDEs due to ease of implementation as compared to analytical approaches.

Over the years various well known numerical methods are developed to solve PDEs such as finite difference, finite element, finite volume, variation iteration method. Almost all traditional numerical techniques require mesh discretization to solve PDEs sometime which is not possible due to the complex domain. This leads to the development of meshfree techniques. Meshfree methods are independent of the generation of mesh. Radial Basis Function (RBF) methods are such meshfree techniques which are very popular due to their meshless nature and ease of implementation.

Different RBF methods are developed over the years [1] for the solution of different types of PDEs. Fasshauer [2] established a new approach studied as radial basis function based pseudospectral method (RBF-PS). In RBF-PS method, RBF is combined with the highly accurate pseudospectral approach. Uddin and Ali [3] used this hybrid technique to solve a non-linear

 

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PDE. Uddin then applied this approach to find the numerical solution of Boussinesq and equal width equation.[4-5]. A two dimensional telegraph equation is solved recently by Abbasbandy et al. [6] using RBF-PS method. Recently, Arora and Bhatia [7] applied this technique to solve one and two dimensional Fisher's equation.

Consider the non-linear parabolic PDE of the form

$$u_t = \mu u_{xx} + \alpha u + \beta u^{\gamma} + \psi(x,t,u,u_x) \qquad (1)$$

where$\mu, \alpha, \beta$ and $\gamma$ are real constants. Various well known models in fluid dynamics, mathematical biology and image processing and plasma physics are modelled with the help of equation (1). For $\mu = 1, \alpha = -4, \beta = 4$and $\gamma = 3$, the equation reduces to well-known Allen-Cahn equation. Similarly for $\mu = 1, \alpha = -\beta$and $\gamma = 2$, equation (1) reduces to Fisher's equation. The Newell-whitehead-Segal (NWS) equation can be obtained by replacing $\beta\ to - \beta$ and$\gamma = 2$, so equation (1) takes the form

$$u_t = \mu u_{xx} + \alpha u - \beta u^2 + \psi(x,t,u,u_x) \qquad (2)$$

The NWS equation is mainly used in the study of patterns in fluid mechanics. Researchers applied different techniques to solve NWS equation. Ezzati and Shakibi [8] solved NWS equation with the help of multiquadric quasi-interpolation method. Dehgani et al. applied the homotopy analysis method [9]. Macias-Diaz and Ruiz-Ramirez [10] applied non-standard symmetry-preserving method to solve the generalized form of NWS equation. S. Mishra et al. [11] applied $\frac{G'}{G}$ expansion to fine the exact solution of the equation. Saravanan and Magesh [12] compared thwo different numerical techniques for the solution of NWS equation. Zahra et al. [13] applied Cubic B-spline collocation method to solve the NWS equation.

In the current work, the RBF-PS method is proposed for the numerical simulation of NWS equation (2) with the boundary conditions

$$u(a,t) = f_1(t),\ u(b,t) = f_2(t),\ \ t \in [0,T] \qquad (3)$$

and the initial condition $u(x,0) = g(x),\ x \in [a,b]$ (4)

Inthis hybrid method, the given NWS equation is discretized with the use of RBF and reducedto a system of ODEsthen the reduced system of ODEs are solved by an appropriate ODE solver in MATLAB.

## 2. Implementation of the RBF-PS method for NWS equation:

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In this segment, the RBF-PS method is applied for solving NWS equation (2) with initial (4) and boundary condition (3) by approximating the space derivatives with the help of RBFs. The whole domain [a, b] is divided into points$x_k = 1,2,3, \ldots N$. The RBF approximation for $u(x,t)$ can be written in the form as

$u_N = \sum_{k=1}^{N} \zeta_k \phi_k(\|x - x_k\|)$ (5)

where$\phi_k = \phi(r)\ and\ r = \|x - x_k\|$ denotes the Euclidean distance between the points $x$ and $x_k$. As $\phi_k$ is the radial basis function. To satisfy the boundary condition,we need to find the basis function which satisfies it.

Equation (5) at evaluated at various nodes$x_k = 1,2,3, \ldots N$, we get

$u_N(x_i) = \sum_{k=1}^{N} \zeta_k \phi_k(\|x_i - x_k\|)$ (6)

In thematrix form, we can write equation (6) as

$U = AC$ (7)

where$A_{ik} = \phi_k(\|x_i - x_k\|)$ are the radial basis functions at nodes and $\Upsilon = [\zeta_1, \zeta_2, \ldots, \zeta_N]^T$ are the unknown interpolation coefficients.

Now, the derivative of $u_N$of (6) by differentiating the basis functions, as

$\frac{d}{dx_i} u_N(x_i) = \sum_{k=1}^{N} \zeta_k \frac{d}{dx_i} \phi_k(\|x_i - x_k\|)$ (8)

Again, evaluate (8) at the grid points,$x_k = 1,2,3, \ldots N$, we get

$U_x = A_x C$ (9)

where the entries of the derivative matrix $A_x$ are $\frac{d}{dx} \phi_k(\|x_i - x_k\|)$.

The condition that the evaluation matrix A in (7) is invertible, depends on various factors like RBF selected and the chosen grid points. The matrix generated by using a positive definite RBFs is always lead to a non-singular matric and hence invertible.

Since A is invertible so from equation (7) $C = A^{-1}U$ and equation (9) becomes

$U_x = A_x A^{-1} U = D_x U$ (10)

where$D_x$is known as Differentiation matrix

Similarly, one can find the differentiation matrix concerning the second and higher order derivatives, i.e

$U_{xx} = A_{xx} A^{-1} U = D_{xx} U$ (11)

Now discretize the NWS equation in space at the collocation points $x_k = 1,2,3, \ldots N$

$$\frac{du}{dt} = \mu u_{xx} + \alpha u - \beta u^2 + \psi(x,t,u,u_x)$$

 

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Substitute (10 and (11) in the above equation and we get

$$\frac{dU_N}{dt} = \mu D_{xx} U_N + \alpha U_N - \beta U_N * U_N + \psi(x, t, U_N, D_x U_N) (10)$$

By RBF-PS scheme, equation (2) reduces to a system of ODEs. The obtained ODEs can be discretized in time using any ODE solver like ode113, ode45 from MATLAB. We have used ode45 ODE solver to solve the resultant ODEs.

## 3. Numerical Results:

In this section, we have applied the RBF-PS method for the numerical simulation of NWS equation. To establish the accord and efficiency of the method, the root mean squared (RMS) error and the maximum absolute error of the approximations are reported which is defined as

$$\text{RMS error} = \sqrt{\frac{\sum_{k=1}^{N} (U(x_k, t_n) - U_N(x_k, t_n))^2}{N}}$$

$$L_\infty \text{error} = \max_{1 \le k \le N} |U(x_k, t_n) - U_N(x_k, t_n)|$$

for $0 \le t_n \le T$, where $U$ and $U_N$ denotes the exact and the numerical solutions. In this paper, we used a positive definite Cubic Matérn RBF given by $\varphi(r) = (15 + 15\epsilon r + 6(\epsilon r)^2 + (\epsilon r)^3) e^{-\epsilon r}$ as a basis function for approximation.

**Example 3.1** Consider the NWS equation (2) over the domain [0,1] with$\mu = 1, \alpha = 1, \beta = 1 \; and \;\; \psi(x, t, u, u_x) = 0$.

The exact solution of the equation is given by

$$u(x, t) = \left(1 + exp(x/\sqrt{6} - 5/6t)\right)^{-2}$$

The equation is solved with initial and boundary conditions, extracted from the exact solution.

Table 1 represents various error norms for different values of t. Error norms are calculated with N=21 and $\Delta t = 0.001$. The efficiency of the method is checked by calculating the absolute error for different time levels. Figure 9 represents the performance of the solution graphically with $\Delta t$=0.001, N=21 for t $\le$ 1.

***Table 1.*** *The error norm withshape parameter 0.401232 and* $\Delta t = 0.001$ *at some values of t for example3.1*

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| Time (t) | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|
| $L_\infty$ | 6.8217E-06 | 3.1191E-04 | 8.8448E-04 | 1.8000E-04 |
| $L_2$ | 2.0943E-04 | 8.0171E-04 | 1.7000E-04 | 3.0001E-04 |
| **RMS** | 4.5701E-05 | 1.7495E-04 | 3.8049E-04 | 6.4594E-04 |

***Table 2.*** *Point wise absolute error calculated for the obtained solutions at different values of t for example 3.1*

| ***x/t*** | ***0.2*** | ***0.4*** | ***0.6*** | ***0.8*** | ***1*** |
|---|---|---|---|---|---|
| ***0.2*** | 0.000349017 | 0.001284273 | 0.002603606 | 0.004159367 | 0.005824692 |
| ***0.3*** | 0.000231493 | 0.001021764 | 0.002183953 | 0.003612926 | 0.005185501 |
| ***0.4*** | 0.000142901 | 0.000808174 | 0.001829434 | 0.003142701 | 0.00463048 |
| ***0.5*** | 7.8436E-05 | 0.000637923 | 0.00153556 | 0.002745487 | 0.004157566 |
| ***0.6*** | 4.24454E-05 | 0.000512637 | 0.001310943 | 0.00243526 | 0.00378554 |
| ***0.7*** | 1.8904E-05 | 0.000417711 | 0.001131747 | 0.002182421 | 0.003479726 |
| ***0.8*** | 6.82169E-06 | 0.000350564 | 0.000996679 | 0.001987124 | 0.003241617 |
| ***0.9*** | 1.88234E-05 | 0.000319572 | 0.000924964 | 0.001876565 | 0.003105548 |

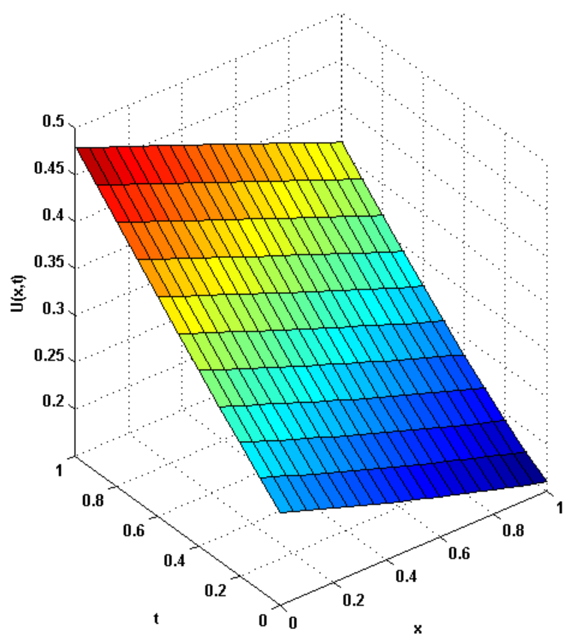


***Figure 1:*** *Numerical solution of Example3.1 with Δt=0.1, N=21 for t≤1*

**Example 3.2**Consider the NWS equation (1) over the domain [0,1] with$\mu = 1, \alpha = 1, \beta = -1\,, \gamma = 4\ and\ \ \psi(x,t,u,u_x) = 0$.

The exact solution of the equation is given by

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$$u(x,t) = \left(\frac{1}{2}\tanh\left(\frac{-3}{2\sqrt{10}}\left(x - \frac{7}{\sqrt{10}}t\right) + \frac{1}{2}\right)\right)^{\frac{2}{3}}$$

The equation is solved with initial and boundary conditions, extracted from the exact solution.

Table 3 represents various error norms for different values of t. The efficiency of the method is checked by calculating the absolute error for different time levels and represented by the table 4. Figure 2presents the performance of the solution graphically with Δt=0.001, N=21 for t ≤ 1.

***Table 3.*** *The error norm with shape parameter =0.401232 and* $\Delta t = 0.0001$ *with no. of collocation points =21 at different t for Example3.2*

| **Time (t)** | **0.001** | **0.005** | **0.009** |
|---|---|---|---|
| $L_\infty$ | 2.3419E-04 | 1.200E-04 | 2.100E-04 |
| $L_2$ | 1.400E-03 | 7.000E-03 | 1.260E-02 |
| **RMS** | 3.0569E-04 | 1.500E-03 | 2.100E-03 |

***Table 4.*** *Absolute error calculated for the obtained solutions at different time-levels for different values of x in Example 3.2*

| ***x/t*** | ***0.001*** | | ***0.005*** | | ***0.009*** |
|---|---|---|---|---|---|
| ***0.2*** | 0.000363245 | | 0.001811836 | | 0.003258663 |
| ***0.3*** | 0.00043965 | | 0.002196863 | | 0.003958422 |
| ***0.4*** | 0.000529101 | | 0.002647729 | | 0.004780554 |
| ***0.5*** | 0.000637593 | | 0.003198606 | | 0.005792947 |
| ***0.6*** | 0.000777131 | | 0.003918582 | | 0.007148547 |
| ***0.7*** | 0.000974419 | | 0.00497648 | | 0.009240872 |
| ***0.8*** | 0.001303793 | | 0.006899552 | | 0.013120429 |
| ***0.9*** | 0.002072925 | | 0.011270976 | | 0.020980784 |

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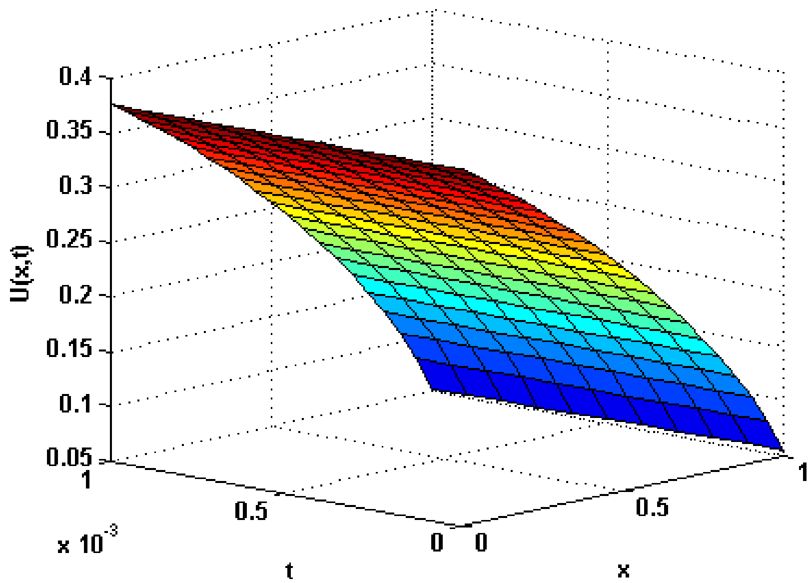


***Figure 2:*** *Numerical solution of Example3.2 with Δt=0.0001, N=21 for t≤0.001*

## 4. Conclusion

A radial basis function pseudospectral approach is presented for the numerical solution of NWS equation. The implementation of the method is very simple and easy. For discretizing the spatial derivatives, radial basis function differentiation matrices are used. RBF-PS method transforms the given equation into a system of ODEs. It is very easy to solve the ODEs with any ODE solver. We use ODE45 to solve the ODEs.Implementation of the method is presented with numerical examples. Various error norms are used to check the accord of the method. For validation, numerical examples are presented. It is found that the present scheme provides an alternative method to solve the NWS equation.

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