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# Flatness of curvature tensors on a Kählerian manifold

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## ***Abstract:***

*In this paper, we have studied the flatness of different type of curvature tensors. We have obtained the conditioned under which the flatness of projective tensor, conformal tensor, concircular tensor and conharmonic tensor with quarter symmetric connection occur. In this paper group manifold has been discussed.*

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## 1. Introduction

In 1975, a linear connection is introduced by S. Golab [8] called quarter-symmetric connection. Golab [8] defined that condition $T(X,Y) = \omega(Y)\varphi X\ \omega(X)\varphi Y$, for tensor field $\varphi$ of type (1,1) and $X, Y$ are arbitrary vector fields. A linear connection is a non-metric connection if covariant derivative of the Riemannian metric $g$ does not vanish i.e. $\nabla g \neq 0$.

The semi-symmetric connection under non-metric condition has been studied in [6] (2008) in a Kähler manifold. Recently, some results have been obtained in weakly symmetric manifold in [5] (2017). Further they studied semi-symmetric non-metric connection [10] (2017) and quarter-symmetric non-metric connection [9] (2018) on lift of Kähler manifolds [10]. Nijenhuis tensor becomes zero in a Kähler manifold [6]. In the same paper, they [6] obtained some results related to contra-variant vector field in Kähler manifold with semi-symmetric connection. Flatness of Kähler manifold under weak symmetric condition has been discussed in [1] (2014). In 2015 a new type of results has been studied [2] in an almost Hermitian manifold. The same authors [3] further studied a special type of Kähler manifold under weak symmetric condition. The study of conformal connection has been extended by Chaturvedi and Pandey [4] in an almost Hermitian manifold in 2016.

Let $M$ be differentiable manifold of dimension 2k, for non-negative integer k. If the conditions

$$F^2\,(X) + X = 0,\ g(FX, FY) = g(X, Y), (\nabla_X F)Y = 0, \tag{1.1}$$

hold then M becomes Kähler manifold.

In this paper, we have considered a connection $\nabla^*$

$$\nabla^*_X Y = \nabla_X Y + \omega(Y)FX \tag{1.2}$$

satisfying

$$(\nabla^*_X g)(Y, Z) = \alpha[\omega(Y)g(FX, Z) + \omega(Z)g(FX, Y) \tag{1.3}$$

$$T^*(X, Y) = \omega(Y)FX - \omega(X)FY \tag{1.4}$$

respectively for $\omega$ 1-form defined by $\omega(X) = g(X, \rho)$, where $\rho$ is an associated vector field.

## 2. Preliminaries

Let $M$ denotes a manifold of dimension 2k, for non negative integer, then Riemannian tensor is defined by

$$R(A, B, P) = \nabla_A \nabla_B P - \nabla_B \nabla_A P - \nabla_{[A,B]} Z \tag{2.1}$$

and the Ricci tensor is contraction of $R$.

Now, equation (2.1) becomes for $\nabla^*$

$$R^*(X, Y, Z) = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z \tag{2.2}$$

Using (1.2) in (2.2), we get

$$R^*(A, B, P) = R(A, B, P) + [(\nabla_A \omega)(P)FB - (\nabla_B \omega)(P)FA] + [(\nabla_A F)B - (\nabla_B F)A + \omega(FB)FA - \omega(FA)FB]\omega(P). \tag{2.3}$$

If the associated vector field is unit parallel then $\nabla_X \rho = 0$, which imply

$$(\nabla_P \omega)(Z) = 0. \tag{2.4}$$

Now, using (1.1) and (2.4) in (2.3), we get

$$R^*(A, B, P) = R(A, B, P) + [\omega(FB)FA - \omega(FA)FB]\omega(P). \tag{2.5}$$

Further, contracting (2.5), we get

$$S^*(B, P) = S(B, P) + \omega(B)\omega(P) \tag{2.6}$$

Again, contracting (2.6), we get

$$r^* = r \tag{2.7}$$

## 3. Condition on flatness of projective tensor

The projective tensor is defined by

$$W(A, P, K) = R(A, P, K) - \frac{1}{n-1}[S(P, K)A - S(A, K)P] = 0. \tag{3.1}$$

For projective flat manifold, we get

$$R(A, P, K) = \frac{1}{n-1}[S(P, K)A - S(A, K)P]. \tag{3.2}$$

Now, the projective curvature tensor for quarter symmetric connection $\nabla^*$ defined in (1.2) is given by

$$W^*(A, P, K) = R^*(A, P, K) - \frac{1}{n-1}[S^*(P, K)A - S^*(A, K)P]. \tag{3.3}$$

Using (2.5), (2.6) and (3.1) in (3.3), we get

$$W^*(A, P, K) = W(A, P, K) + \left[\omega(FP)FA - \frac{1}{n-1}\omega(P)A - \omega(FA)FP + \frac{1}{n-1}\omega(A)P\right]\omega(K). \tag{3.4}$$

Hence for projective flat manifold we get

$$\omega(FP)FA - \omega(FA)FP = \frac{1}{n-1}[\omega(P)A - \omega(A)P]. \tag{3.5}$$

Replacing $A$ and $P$ by $FA$ and $FP$ respectively in (3.5), we get

$$\omega(P)A - \omega(A)P = \frac{1}{n-1}[\omega(FP)FA - \omega(FA)FP]. \tag{3.6}$$

Subtracting (3.6) from (3.5), we get

$$[\omega(FP)FA-\omega(FA)FP-\omega(P)A+\omega(A)P](n-2)=0. \tag{3.7}$$

Let us defined

$$D(A,P)=\omega(FP)FA-\omega(P)A. \tag{3.8}$$

With the help of (3.8), equation (3.7) yields

$$[D(A,P)-D(P,A)](n-2)=0. \tag{3.9}$$

Equation (3.9) implies either manifold has dimension two or

$$D(A,P)=D(P,A). \tag{3.10}$$

From above, we conclude that

***Theorem 3.1:*** *For M being a Kähler manifold and $\nabla^*$ defined by (1.2) then for a associated parallel unit vector field ρ the manifold becomes projective flat if and only if $D(X,Y)$ defined in (3.8) is symmetric or the manifold is of dimension two.*

## 4. Condition on flatness of concircular tensor

The concircular curvature tensor on a manifold of dimension n is defined by

$$C(A,P,K)=R(A,P,K)-\frac{r}{n(n-1)}[g(P,K)A-g(A,K)P. \tag{4.1}$$

The concircular curvature tensor for $\nabla^*$ defined in (1.2) is given by

$$C^*(A,P,K)=R^*(A,P,K)-\frac{r^*}{n(n-1)}[g(P,K)A-g(A,K)P]. \tag{4.2}$$

Using (2.5) in (4.2), we get

$$C^*(A,P,K)=C(A,P,K)+[\omega(FP)FA-\omega(FA)FP]\omega(K). \tag{4.3}$$

For the concircular flat manifold, we get

$$\omega(FP)FA-\omega(FA)FP=0. \tag{4.4}$$

Hence we have

***Theorem 4.1:*** *For M being a Kähler manifold and $\nabla^*$ defined by (1.2) then for a associated parallel unit vector field ρ the manifold becomes concircular flat if and only if (4.4) holds.*

## 5. Condition on flatness of conharmonic tensor

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The conharmonic curvature tensor on a manifold of dimension n is defined by

$$L(A,P,K) = R(A,P,K) - \frac{1}{(n-2)}[S(P,K)A - S(A,K)P - g(A,K)QP + g(P,K)QA]. \tag{5.1}$$

The associated conharmonic curvature tensor is given by

$$L(A,P,K,T) = R(A,P,K,T) - \frac{1}{(n-2)}[S(P,K)(A,T) - S(A,K)(P,T) - g(A,K)S(P,T) + g(P,K)S(A,T). \tag{5.2}$$

Now, the conharmonic tensor with respect to $\nabla^*$ is given by

$$L^*(A,P,K,T) = R^*(A,P,K,T) - \frac{1}{(n-2)}[S^*(P,K)g(A,T) - S^*(A,K)g(P,T) - g(A,K)S^*(P,T) + g(P,K)S^*(A,T)]. \tag{5.3}$$

With the help of (2.5) and (2.6), the equation (5.3) reduces

$$L^*(A,P,K,) = L(A,P,K,T) + [\omega(FP)g(FA,T) - \omega(FA)g(FP,T)]\omega(K) - \frac{1}{(n-2)}[\omega(P)\omega(K)g(A,T) - \omega(A)\omega(K)g(P,T) - \omega(P)\omega(T)g(A,K) + \omega(A)\omega(T)g(P,K)]. \tag{5.4}$$

If the manifold becomes conharmonic flat then

$$[\omega(FP)FA - \omega(FA)FP]\omega(K) = \frac{1}{(n-2)}[\omega(P)\omega(K)A - \omega(A)\omega(K)P - \omega(P)g(A,K)\rho + \omega(A)g(P,K)\rho]. \tag{5.5}$$

Replacing $A$ and $P$ by $FA$ and $FP$ respectively in (5.5), we get

$$[\omega(FP)FA - \omega(FA)FP - \omega(P)A + \omega(A)P](n-1) = 0. \tag{5.6}$$

Since the manifold is even dimensional. Therefore $(n-1) \neq 0$.

Hence, we get the result

***Theorem 5.1:*** *For M being a Kähler manifold and $\nabla^*$ defined by (1.2) then for a associated parallel unit vector field $\rho$ the manifold becomes conharmonic flat if and only if (5.6) holds.*

## 6. Condition on flatness of conformal curvature tensor

The conformal curvature is given by

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$$V(A,P,K,T) = R(A,P,K,T) - \frac{1}{(n-2)}[S(P,K)g(A,T) - S(A,K)g(P,T) - g(A,K)S(P,T) + g(A,K)S(A,T)] + \frac{r}{(n-1)(n-2)}[g(P,K)g(A,T) - g(A,K)g(P,T)]. \quad (6.1)$$

Now, the conformal curvature for $\nabla^*$ defined in (1.2) is given by

$$V^*(A,P,K,T) = R^*(A,P,K,T) - \frac{1}{(n-2)}[S^*(P,K)g(A,T) - S^*(A,K)g(P,T) - S^*(P,T)g(A,K) + S^*(A,T)g(P,K)] + \frac{r}{(n-1)(n-2)}[g(P,K)g(A,T) - g(A,K)g(P,T)]. \quad (6.2)$$

Using (2.5), (2.6), (2.7) and (6.1) in (6.2), we get

$$V^*(A,P,K,T) = V(A,P,K,T) + [\omega(FP)g(FA,T) - \omega(FA)g(FP,T)]\omega(K) - \frac{1}{(n-2)}[\omega(P)\omega(K)g(A,T) - \omega(A)\omega(K)g(P,T) - \omega(P)\omega(T)g(A,K) + \omega(A)\omega(T)g(P,K). \quad (6.3)$$

Let the manifold is conformally flat then we have

$$[\omega(FP)FA - \omega(FA)FP]\omega(K) - \frac{1}{n-2}[\omega(P)\omega(K)A - \omega(A)\omega(K)P - \omega(P)g(A,K)\rho + \omega(A)g(P,K)\rho] = 0. \quad (6.4)$$

Hence, we conclude that

***Theorem 6.1:*** *For M being a Kähler manifold and $\nabla^*$ defined by (1.2) then for a associated parallel unit vector field $\rho$ the manifold becomes conformally flat if and only if (6.4) holds.*

## 7. Group Manifold on a Kähler

For a group manifold, we know that

$$R(A,P,K) = 0 \qquad \text{and} \qquad (\nabla_A S)(P,K) = 0. \quad (7.1)$$

Now, the manifold will be group manifold with respect to $\nabla^*$ if and only if

$$R^*(A,P,K) = 0. \qquad \text{and} \qquad (\nabla_A S^*)(P,K) = 0. \quad (7.2)$$

From (2.5) and (7.2), we get

$$\omega(P)A - \omega(A)P = 0. \quad (7.3)$$

Now, the covariant derivative of (2.6) yields

$$\omega(P)S(FA,K) + \omega(K)S(FA,P) + \omega(P)\omega(K)\omega(FA) = 0. \quad (7.4)$$

Replacing $A$ by $FA$ in (7.4), we get

$$\omega(P)S(A,K)+\omega(K)S(A,P)+\omega(A)\omega(P)\omega(K)=0. \tag{7.5}$$

Hence, we have the result

***Theorem 7.1:*** *For M being a Kähler manifold and $\nabla^*$ defined by (1.2) then for a associated parallel unit vector field ρ then it is a group manifold if and only if (7.3) and (7.5) holds.*

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