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# Regarding Corners of Invo-Clean Unital Rings

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**Abstract**

*We prove that if R is an invo-clean ring and e is an idempotent, then the corner subring eRe is also invo-clean. In particular, if for any $n \in N$ the full matrix $n \times n$ ring $M_n(R)$ is invo-clean, then so is R.*

**Keywords:** Rings, Idempotent.

## Introduction

Everywhere in the text of the current paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0. Our standard terminology and notations are mainly in agreement with [11]. For instance, for such a ring R, the letter J(R) stands for the Jacobson radical of R, Nil(R) for the set of nilpotents in R, and Id(R) for the set of idempotent in R. For some arbitrary but a fixed $e \in Id(R)$, the subrings eRe and (1 - e)R(1 - e) of R are named corner subrings of R. As usual, $M_n(R)$ will denote the full $n \times n$ matrix ring, where $n \in N$.

Imitating [13], a ring is called clean if each its element is the sum of a unit and an idempotent, whereas if this unit is 2-torsion, i.e., it is an involution, the clean ring is called invo-clean in [2] and [3]. On the other vein, a ring is said to be nil-clean in [9] if every its element the sum of a nilpotent and an idempotent, while in [7] and [1] a ring is termed weakly nil-clean if every its element is either the sum or the difference of a nilpotent and an idempotent.

Our motivation in writing up this short article is to show the somewhat surprising property that invo-cleanness is closed for taking corners, although the same property is not longer true for the cleanness as it has been demonstrated in [14]. Our statements are very close to these from [4], [5] and [6].

 

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After that, we will comment some idempotent properties as defined in [6] and we will finish with a characterization of the structure of special sorts of weakly nil-clean rings by extending the corresponding results from [12, Corollary 2.6] and [15, Proposition 4.3], respectively.

## 2. Main Results and Problems

We start here with the following technicality.

**Lemma 2.1.** Suppose that R is a ring. Then R is an invo-clean ring if, and only if, $R = A \times B$, where A is a nil-clean ring such that the quotient $\overline{A} = A/J(A)$ is nil-clean of nilpotency index $\leq 2$, J(R) is nil, the elements of Nil(R) satisfy the equation $a^2 + 2a = 0$, and B is a ring which is either zero or a sub direct product of the fields $Z_3$.

**Proof**. It follows by a direct combination of [2] and [12, Theorem 3.3] (compare also with Lemmas 3.1 and 3.2 from [12]).

The next assertion is our critical tool. As aforementioned it, however, contrasts the counterexample in [14], where a clean ring R was constructed for which eRe is not clean.

**Theorem 2.2.** If R is an invo-clean ring, then eRe is also an invo-clean ring for any idempotent e of R. In addition, for all $n \in N$, if $M_n(R)$ is invo-clean, then so is R.

**Proof**. In virtue of Lemma 2.1, we decompose $R = A \times B$, where A, $\overline{A}$ and B have the same meaning as above. Furthermore, letting $e = (e_1, e_2)$ be an idempotent in R, we detect that $e_1$ is an idempotent in A and $e_2$ is an idempotent in B. Therefore, a plain check shows that $eRe = (e_1Ae_1)\times(e_2Be_2)$. It is obvious that the second direct factor $e_2Be_2$ is either zero or a subdirect product of the $Z_3$'s. This is true, because B being commutative of characteristic 3 ensures that all its elements satisfy the equation $x^3 = x$ and so the elements of the corner ring $e_2Be_2 = e_2B$ satisfy that equation too.

Now, setting $S = e_1Ae_1$, we deduce with the aid of [11] that $J(S) = e_1J(A)e_1 = S \cap J(A) \subseteq J(A)$, whence J(S) is nil as well, and that $Nil(S) \subseteq Nil(A)$. However, $S/J(S) = e_1Ae_1/[(e_1Ae_1) \cap J(A)] \cong (e_1Ae_1+J(A))/J(A) = \overline{e_1}\ \overline{A}\ \overline{e_1}$, where $\overline{e_1} = e_1 + J(A)$ and the last equality is fulfilled because of the formula $(e + J(A))(a + J(A))(e + J(A)) = eae + J(A)$, which holds for any $a \in A$. Exploiting furthermore [12, Corollary 2.5], we obtain that $\overline{e_1}\ \overline{A}\ \overline{e_1}$ is nil-clean of index of nilpotence at most 2. Hence again Lemma 2.1 is in use to conclude that eRe is an invo-clean ring, as expected.

As for the second part-half, it is well known that $R \cong fM_n(R)f$ for some idempotent $f \in M_n(R)$. Thus the first part-half directly applies to get the desired implication.

We will now comments some idempotent properties in arbitrary rings.

**Proposition 2.3.** Suppose R is a ring and $e \in Id(R)$. Then the following are equivalent:

 

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(1) e is central, that is, er = re for all $r \in R$.

(2) ef = fe for all $f \in Id(R)$.

(3) efe = fef for all $f \in Id(R)$, that is, e is symmetric.

(4) ef = fef (resp., fe = fef) for all $f \in id(R)$, that is, e is left (resp., right) regular.

**Proof**. Given the truthfulness of (3), for any $r \in R$ we consider the element e + er(1 - e). A routine check shows that it is an idempotent of R, say f.

We observe also that fe = e as well as that ef = f. Therefore, efe = fef yields that fe = ef, i.e., e = f. Consequently, f = e + er(1 - e) implies that er(1-e) = 0 and so that er = ere. Similarly, (1-e)re = 0 and thus re = ere. Finally, er = re, as required. This proves the equivalence (1) $\leftrightarrow$ (3). The other points (2) and (4) can be handled by analogy. In view of the last claim, we may state:

*Remark 2.4.* In [6, Definitions 1.1 and 1.2] the concepts of symmetric and left-right regular idempotents were introduced. In accordance with Proposition 2.3, the word "differ", which is stated on line 4 after Definition 1.2 referred to above, surely should be replaced by the phrase "eventually differ".

Let us recall now that an idempotent e of a ring R is said to be left-central if, for each $r \in R$, the equality re = ere holds, and it is said to be right-central provided er = ere holds. Somewhere in the existing literature this idempotent is called semicentral, left or right respectively. It is clear that if e is simultaneously left-central and right-central, then it is central, i.e., er = re for every $r \in R$.

If in a way of similarity we consider the left equality re = rer and its right analogue er = rer, it must be that $(re)^2$ = re and $(er)^2$ = ere as well as that $(er)^2$ = er and $(re)^2$ = ere.

What is of some interest is to know what these conditions on the idempotent e will imply. In particular, what we can discover when $(er)^2 = (re)^2$ in the general case not provided the validity of the previous equalities for e?

We now will be involved with the structure of weakly nil-clean rings of bounded nilpotency index. Consulting with [15, Definition 2.3], we shall say that a weakly nil-clean ring R has a finite nil index if it exists $k \in N$ which is the least integer such that $(r - e)^k = 0$ or $(r + e)^k = 0$ for every $r \in R$ and some $e \in Id(R)$ which depends on r.

Let us recall that a ring R is called strongly $\pi$-regular, provided for each $a \in R$ there is $n \in N$ with the property that $a^n \in a^{n+1}R \cap Ra^{n+1}$.

Specifically, the following strengthening of [15, Proposition 4.3] and [12, Corollary 2.6] holds:

 

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**Proposition 2.5.** If R is a weakly nil-clean ring of nil index not exceeding 2, then R is strongly $\pi$-regular.

**Proof**. Our first assertion is that R has a bounded by 2 index of nilpotence. In fact, if t is an arbitrary nilpotent in R, then we can write that t = e + q or t = -e+q, where $q^2 = 0$. Consequently, one writes that e = t-q or e = q – t and thus it follows from [10, Proposition 2] that e = 0, so that t = q sustaining our claim.

On the other side, it was proved in [1] that weakly nil-clean rings are clean and thus exchange. Moreover, it was also shown there that every homomorphic image of R has nil Jacobson radical. Henceforth, we are now in a position to employ [8, Lemma 3.9] inferring that R is strongly $\pi$-regular, indeed.

We end our work with the following two questions of interest.

**Problem 2.6.** Let R be a ring in which 2 is nilpotent, and let e ∈ R be an idempotent. If both eRe and (1 - e)R(1 - e) are invo-clean rings, does it follow that R is also invo-clean?

In this aspect, let us comment that the matrix ring $M_n(Z_3)$ over the invo-clean ring $Z_3$ is not invo-clean as well since, in conjunction with [2] or [3], all invo-clean rings of characteristic 3 must be commutative (compare also with [6]). Moreover, although all units in $Z_3$ are involutions, this is not the case in the matrix ring $M_2(Z_3)$ over $Z_3$ even in the case n = 2. As an illustration of this, the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has the property that its square $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^2$ is equal to $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. That is why, a ring R is said to be almost invo-clean if, for every element r in R, we have that r = v + e, where $e^2 = e$ and $v^2 = 1$ or $v^2 = -1$, calling such an element v an almost involution. Note that our matrix illustrated above is, actually, an almost involution. So, one may ask of whether or not Mn(R) is an almost invo-clean ring, provided that R is an invo-clean ring. As a starting point of view, this could be firstly checked for $M_n(Z_3)$.

**Problem 2.7**. If R is a ring of index of nilpotence at most 2 and eRe and (1-e)R(1-e) are both nil-clean rings of index of nilpotence not exceeding 2, is R also nil-clean?

In case that is true, it will improve on [4, Lemma 2.3], where both corners are assumed to be boolean rings.

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