ISSN: 0971-1260
Vol-22-Issue-10- November-2019

Structure Of The Nil Graphs For Product Of Rings

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ABSTRACT :

Let the set of nilpotent elements of the ring R and the n × n matrix ring over R are Nil(R) and $M_n(R)$ respectively. Here, we have taken a simple, undirected graph. For graph G, the set of vertices and the set of edges are indicated by V(G) and E(G) respectively. In this paper, we study the structure of the nil graph $\Gamma_N(R)$, where $R = \mathbb{Z}_m \times \mathbb{Z}_n$ and determine the chromatic number of the graph $\Gamma_N(\mathbb{Z}_m \times \mathbb{Z}_n)$, for different values of *m* and *n*.

Keyword: Nil graph, Chromatic number, Commutative Ring.

1.INTRODUCTION

The set of nilpotent elements of the ring R and the n × n matrix ring over R are indicated by Nil(R) and $M_n(R)$ respectively. Here, by graphic, we say a simple, undirected graph. For graph G, the set of vertices and the set of edges are indicated by V(G) and E(G) respectively.
For a positive integer, P. Grimaldi [1] has defined and researched the different properties of the unit graph G($\mathbb{Z}_n$), of the integer ring modulo n, with a vertex set $\mathbb{Z}_n$ and two distinct vertexes adjacent if and only if their sum is a unit. Further in [2] authors have generalized G($\mathbb{Z}_n$) to G(R) graph unit, where R is an arbitrary associative ring with a non-zero identity. We introduced the nilpotent graph G(R) associated with the finite ring R. The nilpotent graph G(R) of the ring R is defined by R \ Nil(R) as the vertex set and two vertices x and y are adjacent if and only if x + y is the nilpotent element in R.

P.W Chen [3] proposed a certain kind of graph structure on a commutative ring R by including the nil-elements of R.

The main object of their paper was to study the ring theoretic properties of R and graph theoretic properties of $\Gamma_N(R)$. The main focus of their work was on non-reduced rings since

ISSN: 0971-1260
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for a reduced ring the graph $\Gamma_N(R)$ is actually the general zero-divisor graph. They proved that for a non-reduced commutative ring the graph $\Gamma_N(R)$is connected, the diameter of $\Gamma_N(R)$ is less than or equal to 2 and the girth of $\Gamma_N(R)$is 3 or ∞.

M. J. Nikmehr and S. Khojasteh [4] continued working on the modified definition of A. H. Li and Q. S. Li [5] and named the graph as Nilpotent graph and classified all finite rings whose nilpotent graphs have diameter at most three.

For a ring R , we denote by $M_n(R)$, R^n and I , the ring of all $n \times n$ matrices, the set of $n \times 1$ matrices over R and the identity matrix, respectively. Also, for any i and j , $1 \le i, j \le n$, we use E_{ij} to denote the element of $M_n(R)$ whose (i, j) -entry is 1 and other entries are 0 .

We represent the Jacobson radical of R by J (R). When its Jacobson radical is zero, a ring R is called semisimple. We know from the Wedderburn-Artin that any semi-simple Artinian ring isomorphic to the direct product of finitely many complete matrix rings over divisional rings. If R has no non-zero nilpotent elements, a Ring R is called reduced.

In [5], the authors proved that if R is a left Artinian ring, then $\mathrm{diam}(\Gamma_N(R)) \le 3$. In Section 2, we prove that if F is a field and $n \ge 3$, then $\mathrm{diam}(\Gamma_N(M_n(F))) = 2$. If F is a field, we show that $\mathrm{diam}(\Gamma_N(M_2(F))) = 3$. In addition, we propose a new proof for [105] Theorem 2.3 when R is a finite ring. We can also define the diameter of the nilpotent graph with J (R) = 0. for all finite rings.

Here, we study the nil graphs of the ring $\mathbb{Z}_m \times \mathbb{Z}_n$, and determine the chromatic number of the said graph.

2. THE STRUCTURE OF $\Gamma_N(\mathbb{Z}_m \times \mathbb{Z}_n)$:

The aim of this section is to characterize the adjacency relation between the vertices of the graph $\Gamma_N(\mathbb{Z}_m \times \mathbb{Z}_n)$.

Let (x_1, y_1) and (x_2, y_2) be any two vertices $\Gamma_N(\mathbb{Z}_m \times \mathbb{Z}_n)$. Then (x_1, y_1) and (x_2, y_2) are adjacent to each other if

$(x_1, y_1).(x_2, y_2) \in N(\mathbb{Z}_m \times \mathbb{Z}_n)$

i.e, if $(x_1.x_2, y_1.y_2) \in N(\mathbb{Z}_m \times \mathbb{Z}_n)$

i.e, if $x_1.x_2 \in N(\mathbb{Z}_m)$ and $y_1.y_2 \in N(\mathbb{Z}_n)$

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i.e, if $(x_1x_2)^2 \equiv 0(mod\ m)$ and $(y_1y_2)^2 \equiv 0(mod\ n)$.

Example : Let us take the ring R = $\mathbb{Z}_{25} \times \mathbb{Z}_{18}$. Then

$$
\begin{aligned}
N\left(\mathbb{Z}_{25} \times \mathbb{Z}_{18}\right) &= N\left(\mathbb{Z}_{25}\right) \times N\left(\mathbb{Z}_{18}\right) \\
&= \{0,5,10,15,20\} \times \{0,6,12\} \\
&= \left\{ \begin{matrix} (0,0),(0,6),(0,12),(5,0)(5,6),(5,12),(10,0),(10,6),(10,12) \\ (15,0),(15,6),(15,12),(20,0),(20,6),(20,12) \end{matrix} \right\}
\end{aligned}
$$

3. THE NIL GRAPH $\Gamma_N(\mathbb{Z}_m \times \mathbb{Z}_n)$:

In this section we determine the chromatic number of the graph $\Gamma_N(\mathbb{Z}_m \times \mathbb{Z}_n)$, for different values of *m* and *n*.

Theorem 3.1: Let $\Gamma_N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}\right)$ be the nil graph of the commutative ring $\Gamma_N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}\right)$,where *p* and *q* are distinct primes, then $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}\right)\right) = pq+1$

Proof: Let (x_1, y_1) and (x_2, y_2) $\in V\left(\Gamma_N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}\right)\right)$ be any two distinct vertices. Then $(x_1, y_1).(x_2, y_2) = (x_1x_2, y_1y_2) \in N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2}\right)$ if $(x_1x_2)^2 \equiv 0\left(\mathrm{mod}\ p^2\right)$ and $(y_1y_2)^2 \equiv 0\left(\mathrm{mod}\, q^2\right)$i.e. $p \mid x_1x_2$ and $q \mid y_1y_2$.

The vertex set of the graph can be partitioned into the following subsets.

$A_1 = \{(mp, m'q): p \nmid m\ and\ q \nmid m', 1 \le m \le p-1, 1 \le m' \le q-1$

All elements of A_1 are adjacent.

$A_2 = \{(m, m'q): p \nmid m\ and\ q \nmid m', 1 \le m \le p^2-1, 1 \le m' \le q-1\}$

All elements of A_2 are not adjacent.

$A_3 = \{(mp, m'): p \nmid m\ and\ q \nmid m', 1 \le m \le p-1, 1 \le m' \le q^2-1\}$

All elements of A_3 are not adjacent.

$A_4 = \{(mp, 0): p \nmid m\ , 1 \le m \le p-1\}$. All the elements of A_4 are adjacent.

$A_5 = \{(0, m'q): q \nmid m'\ , 1 \le m' \le q-1\}$. All the elements of A_5 are adjacent.

$A_6 = \{(m, m'): p \nmid m\ and\ q \nmid m', 1 \le m \le p^2-1, 1 \le m' \le q^2-1\}$

All elements of A_6 are not adjacent.

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All elements of the set $A_1 \cup A_4 \cup A_5$ are adjacent to each other and they form a complete subgraph of order $(p-1)(q-1)+p-1+q-1=pq-1$.

The elements of the set A_2, and A_3 are adjacent with each other and are also adjacent to the complete subgraph of order $pq-1$. Therefore the $pq-1$ elements along with one element from A_2, and one element from A_3 together will form a clique of order $pq-1+1+1=pq+1$ *in* $\Gamma_N\left(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}\right)$. So $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}\right)\right)\geq pq+1$.

The elements of the clique can be colored with $pq+1$ colors. The elements of the set A_2 are not adjacent among themselves and so can be colored with same color. Also the elements of the set A_3 can be assigned the same color as they are not adjacent to each other. The elements of the set A_6 are only adjacent to the elements of the set A_1.So we color these elements with the color assigned to the other elements of the graph.

So $\chi\left(\Gamma_N(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})\right)\leq pq+1$. Hence So $\chi\left(\Gamma_N(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2})\right)=pq+1$.

Theorem 3.2: Let $\Gamma_N\left(\mathbb{Z}_{p^2}\times\mathbb{Z}_{p^2q}\right)$ be the nil graph of the commutative ring $\mathbb{Z}_{p^2}\times\mathbb{Z}_{p^2q}$, where *p* and *q* are distinct primes, then $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^2}\times\mathbb{Z}_{q^2}\right)\right)=p^2+2$.

Proof: Let (x_1, y_1) and $(x_2, y_2) \in V\left(\Gamma_N\left(\mathbb{Z}_{p^2}\times\mathbb{Z}_{p^2q}\right)\right)$ be any two distinct vertices. Then $(x_1,y_1)(x_2,y_2)=(x_1x_2y_1y_2)\in N\left(\mathbb{Z}_{p^2}\times\mathbb{Z}_{p^2q}\right)$ *if* $(x_1x_2)^2\equiv 0\left(\bmod p^2\right)$ and $(y_1y_2)^2\equiv 0\left(\bmod p^2q\right)$ i.e. if $p\mid x_1x_2$ *and* $pq\mid y_1y_2$

The vertex set of the graph can be partitioned into the following subsets

$A_1=\{(mp, m'pq): p\nmid m \text{ and } p,q\nmid m', 1\leq m\leq p-1, 1\leq m'\leq p-1\}$

All elements of A_1 are adjacent.

$A_2=\{(mp, m'p): p\nmid m \text{ and } p,q\nmid m', 1\leq m\leq p-1, 1\leq m'\leq pq-1\}$

All elements of A_2, are not adjacent.

$A_3=\{(mp, m'q): p\nmid m \text{ and } p,q\nmid m', 1\leq m\leq p-1, 1\leq m'\leq p^2-1\}$

All elements of A_3 are not adjacent.

$A_4=\{(mp,0): p\nmid m, 1\leq m\leq p-1\}$.

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All elements of A_4 are adjacent.

$A_5 = \{(0, m^{/}pq): p, q \nmid m^{/}, 1 \le m^{/} \le p-1\}$.

All elements of the set A_5 are adjacent.

$A_6 = \{(m, m^{/}pq): p \nmid m, p, q \nmid m^{/}, 1 \le m \le p^2-1, 1 \le m^{/} \le p-1\}$

All elements of the set A_6 are adjacent.

$A_7 = \{(mp, m^{/}): p \nmid m \text{ and } p, q \nmid m^{/}, 1 \le m \le p-1, 1 \le m^{/} \le p^2q-1\}$

All elements of the set A_7 are adjacent.

$A_8 = \{(m, m^{/}q): p \nmid m \text{ and } p, q \nmid m^{/}, 1 \le m \le p-1, 1 \le m^{/} \le p^2-1\}$

All elements of the set A_8 are adjacent.

$A_9 = \{(m, m^{/}p): p \nmid m \text{ and } q \nmid m^{/}, 1 \le m \le p^2-1, 1 \le m^{/} \le pq-1\}$

All elements of the set A_9 are adjacent.

$A_{10} = \{(m, m^{/}): p \nmid m \text{ and } p, q \nmid m^{/}, 1 \le m \le p^2-1, 1 \le m^{/} \le p^2q-1\}$.

All elements of the set A_{10} are adjacent.

All elements of the set $A_1 \cup A_4 \cup A_5$ are adjacent to each other and they form a complete subgraph of order $(p-1)(p-1)+p-1+p-1=p^2-1$.

The elements of the set A_2, A_3 and A_6 are adjacent with each other and are also adjacent to the complete subgraph of order p^2-1. Therefore the p^2, -1 elements along with one element from each of the set A_2, A_3 and A_6 will together form a clique of order

$p^2-1+1+1=p^2-1+1+1=p^2+2$ in $\Gamma_N(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2q})$

So $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2q}\right)\right) \ge p^2+2$.

The elements of the clique can be colored with p^2+2 colors. The elements of the set A_2, are not adjacent among themselves and so can be colored with same color. Also the elements of the set A_3 can be assigned the same color as they are not adjacent to each other. Similarly the elements of the set A_6 are not adjacent among themselves and so they can be colored with same color. The elements of A_7, A_8, A_9 and A_{10} are not adjacent to every vertex of the clique. Therefore we can color these vertices by taking some colors of the clique. So $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2q}\right)\right) \le p^2+2$. Hence $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2q}\right)\right) = p^2+2$.

Theorem 3.3: Let $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q}\right)$ be the nil graph of the commutative ring $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q}$, where p and q are distinct primes and α_1 and α_2, be any positive integers, then

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(a) $\chi(\Gamma_N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q})) = p^{[3\alpha_1/4]}p^{[3\alpha_2/4]} + 2$, if either $\alpha_1 \equiv 1,2(mod4)$ or $\alpha_2 \equiv 1,2(mod4)$.

(b) $\chi(\Gamma_N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q})) = p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}$, if no $\alpha_i \equiv 1,2(mod4)$.

Proof: The vertex set of the graph $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q}\right)$ can be partitioned into the following subsets

$A_{i,j} = \{(x,y) = (mp^i, m^/p^jq) : p \nmid m, m^/, q \nmid m^/, 1 \le m \le p^{\alpha_1 - i} - 1, 1 \le m^/ \le p^{\alpha_2 - j} - 1\}, 0 \le i \le \alpha_1 - 1, 0 \le j \le \alpha_2 - 1.$

$B_{i,j} = \{(x,y) = (mp^i, m^/p^j) : p \nmid m, m^/, q \nmid m^/, 1 \le m \le p^{\alpha_1 - i} - 1, 1 \le m^/ \le p^{\alpha_2 - j}q - 1\},$

$0 \le i \le \alpha_1 - 1, 0 \le j \le \alpha_2.$

$C_j = \{(x,y) = (0, m^/p^jq) : p, q \nmid m^/, 1 \le m^/ \le p^{\alpha_2 - j} - 1\}, 0 \le j \le \alpha_2 - 1.$

$D_i = \{(x,y) = (mp^i, 0) : p \nmid m, 1 \le m \le p^{\alpha_1 - i} - 1\}, 0 \le i \le \alpha_1 - 1.$

$E_j = \{(x,y) = (0, m^/p^j) : p, q \nmid m^/, 1 \le m^/ \le p^{\alpha_2 - j}q - 1\}, 0 \le j \le \alpha_2.$

Let (x_1, y_1) and (x_2, y_2) be any two vertices of $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q}\right)$ such that $x_1 = m_1p^{i_1}, x_2 = m_1^/p^{i_2}, y_1 = m_2p^{i_1}q$ and $y_2 = m_2^/p^{j_2}q$

$x_1x_2 = m_1p^{i_1}m_1^/p^{i_2} = m_1m_1^/p^{i_1+i_2}$ and

$y_1y_2 = m_2p^{j_1}qm_2^/p^{j_2}q = m_2m_2^/p^{j_1+j_2}q^2.$

Then $(x_1,y_1).(x_2,y_2) \in N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q})$ if $i_1 + i_2 \ge \lceil \alpha_1/2 \rceil$ and $j_1 + j_2 \ge \lceil \alpha_2/2 \rceil$.

i.e, if $(x_1x_2)^2 \equiv 0(mod\ p^{\alpha_1})$ and $(y_1y_2)^2 \equiv 0(mod\ p^{\alpha_2}q)$.

Now we can consider the following cases:

Case1: For the vertex set A_{ij}, let $(x_1, y_1) \in A_{i_1j_1}$ and $(x_2, y_2) \in A_{i_2j_2}$ such that $x_1 \equiv m_1p^{i_1}, x_2 = m_2p^{i_2}$ and $y_1 \equiv m_2p^{j_1}q, y_2 = m_2^/p^{j_2}q$. Now we have the following subcases:

1. For $0 \le i_1, i_2 \le [\alpha_1/4] - 1, 0 \le j_1, j_2 \le [\alpha_2/4] - 1$, $(x_1, y_1)(x_2y_2) \notin N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q}\right)$ as $i_1 + i_2 < [\alpha_1/2]$ and $j_1 + j_2 < [\alpha_2/2]$. So (x_1, y_1) and (x_2, y_2) are not adjacent. Therefore

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the elements of $A_{i,j,} i \le [\alpha_1/4]-1$ and $j \le [\alpha_2/4]-1$ are not adjacent to each other. But these elements are adjacent to the elements of $A_{i^/ j^/}$, such that $i+i^/ \ge [\alpha_2/2]$ and $j+j^/ \ge [\alpha_2/2]$, for some $i^/$ and $j^/$.

2. For $[\alpha_1/4] \le i_1, i_2 \le \alpha_1 - 1,\ [\alpha_2/4] \le j_1, j_2 \le \alpha_2 - 1,$ $(x_1, y_1)(x_2, y_2) \in N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}})$ as $i_1 + i_2 \ge [\alpha_1/2]$ and $j_1 + j_2 \ge [\alpha_2/2]$. Therefore the elements of the set $A_{i^/ j^/}, i \ge (\alpha_1/4), j \ge (\alpha_2/4)$ are adjacent to each other. Also all these elements are adjacent to all elements of $A_{i^/ j^/}, (\alpha_1/4) \le i^/ \le (\alpha_1 - 1), (\alpha_1/4) \le j^/ \le \alpha_2 - 1$. Total number of elements in the set $\cup A_{i,j}, i \ge [\alpha_1/4], j \ge [\alpha_2/4]\ is\ (p^{[3\alpha_1/4]} - 1)(p^{[3\alpha_2/4]} - 1)$.

Case 2 : For the vertex set $B_{i,j}$, let $(x_1, y_1), (x_2, y_2) \in B_{i.j}$.

Then for $0 \le i \le \alpha_1 - 1, 0 \le j \le \alpha_2, (x_1 y_1).(x_2, y_2) \notin N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2} q})$ as $(y_1 y_2)^2 \not\equiv 0(\mathrm{mod}\ p^{\alpha_2} q)$. These elements are adjacent to the elements of the set $A_{i^/ j^/}$ such that $i + i^/ \ge [\alpha_1/2]$ and $j + j^/ \ge [\alpha_2/2]$, for some $i^/$ and $j^/$

Case 3 : For the vertex set C_j, let $(0, y_1) \in C_{j_1}$ and $(0, y_2) \in C_{j_2}$, then we have the following subcases:

(i) For $0 \le j_1, j_2 \le [a_2/4] - 1$, the elements of the sets C_{j_1} and C_{j_2} are not adjacent with each other as $0(\mathrm{mod}\ p^{\alpha_2} q)$ and hence $(0, y_1 y_2) \notin N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2} q})$.

(ii) For $[\alpha_2/4] \le j_1, j_2 \le \alpha_2 - 1$ the elements of the sets C_{J_1} and C_{J_2} are adjacent with each other as $(y_1 y_2)^2 \equiv 0(\mathrm{mod}\ p^{\alpha_2} q)$ and hence $(0, y_1 y_2) \in N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2} q})$. Therefore all elements of the set $C_j, j \ge [\alpha_2/4]$ are adjacent to each other. Total number of elements of $UC_j, j \ge [\alpha_2/4]$ is $(p^{[3\alpha_2/4]} - 1)$. All these elements are adjacent to the elements of sets considered in subcase 1(ii) and also adjacent to the elements of $B_{i^/, j^/}$, if $j^/ + j \ge [\alpha_2/2]$.

Case 4 : For the vertex set D_i, let $(x_1, 0) \in D_{i_1}$ and $(x_2, 0) \in D_{i_2}$, then we have the following subcases:

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(i) For $0 \le i_1, i_2 \le [\alpha_2 / 4]-1$, the elements of the sets D_{j_1} and D_{j_2} are not adjacent with each other since $(x_1 x_2)^2 \not\equiv 0 (\bmod p^{\alpha_1})$ as and hence $(x_1 x_2, 0) \notin N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2} q})$.

(ii) For $[\alpha_1 / 4] \le i_1, i_2 \le \alpha_1 - 1$, the elements of the set D_{j_1} and D_{j_2} are adjacent with each other since $(x_1 x_2)^2 \equiv 0 (\bmod p^{\alpha_1})$ as $i_1 + i_2 \ge [\alpha_1 / 2]$ and hence $(x_1 x_2, 0) \in N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2} q})$. Therefore all elements of the set $D_i, i \ge [\alpha_1 / 4]$ are adjacent to each other and to the elements of the set $D_{j'}, i' \ge [\alpha_2 / 4]$. Total number of elements of $UD_i, i \ge [\alpha_2 / 4]$ is $(p^{[3\alpha_1/4]} - 1)$. These elements are adjacent to all elements of the set $A_{i,j}$ and C_j, for $i \ge [\alpha_2 / 4]$ and $j \ge [\alpha_2 / 4]$ and also adjacent to some elements of $B_{i',j'}$ if $j' + j \ge [\alpha_2 / 2]$

Case 5 : For the vertex set Ej , let $(0, y_1) \in E_{j_1}$ and $(0, y_2) \in E_{j_2}$. Then for $0 \le j_1, j_2 \le \alpha_1 - 1, (0, y_1)(0, y_2) \notin N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2} q})$ as $(y_1 y_2)^2 \not\equiv 0 (\bmod p^{\alpha_2} q)$. The elements of` $E_j, j \ge [\alpha_2 / 4]$ are adjacent to the elements of the sets considered in subcase 1(ii),subcase 3(ii) and subcase 4(ii).

Now considering all cases discussed above, the elements of $A_{i,j}, C_j, D_{i,} i \ge [\alpha_1 / 4], j \ge [\alpha_2 / 4]$ will induce a complete subgraph of order $\left(p^{[3_{\alpha_1}/4]} - 1\right)\left(p^{[3_{\alpha_2}/4]} - 1\right) + p^{[3\alpha_1/4]} - 1 + p^{[3\alpha_2/4]} - 1 = p^{[3\alpha_1/4]} - 1.$

(a) If $\alpha_1 \equiv 1,2 (\bmod 4)$ or $\alpha_2 \equiv 1,2 (\bmod 4)$, then the elements of the set $B_{i,j}, i \ge [\alpha_1 / 4], j \ge [\alpha_2 / 4]$ and t he set $E_j, j \ge [\alpha_2 / 4]$ are not adjacent with each other but are adjacent to very member of the above mentioned complete graph.

Therefore the elements of the sets A_{ij}, C_j, D_j for all $i \ge [\alpha_1 / 4], j \ge [\alpha_2 / 4]$ along with any one element either from the set $B_{i,j}, i \ge [\alpha_1 / 4], j \ge [\alpha_2 / 4]$ or from the set $E_j, j \ge [\alpha_2 / 4]$ will induce a clique of order $p^{[3a_1/4]} p^{[3\alpha_2/4]}$. Again the elements of the set $A_{i,j}, i = [\alpha_1 / 4], j = [\alpha_2 / 4]-1$ are not adjacent among themselves but are adjacent to every vertex of the clique. Again the elements of $A_{i,j}, i = [\alpha_1 / 4]-1, j = [\alpha_2 / 4]$ are not adjacent among themselves but are adjacent to every vertex of the clique and also adjacent to $A_{i,j}, i = [\alpha_1 / 4], j = [\alpha_2 / 4]-1$.

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Thus we must have $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{p^{\alpha_2}q}\right)\geq p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}+2$.

Since all elements of the sets $A_{i,j},C_j,D_j$ for all $i\geq[\alpha_1/4], j\geq[\alpha_2/4]$induce a complete graph of order $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}-1$, so these vertices can be colored with $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}-1$ colors. The elements of the set $B_{i,j}, i\geq[\alpha_1/4], j\geq[\alpha_1/4]$ and the set $E_j, j\geq[\alpha_2/4]$ are adjacent to every member of the complete graph and so we can assign one more color to color these vertices. But allelements of $B_{i,j}$ and E_j are not adjacent to each other, so they can be colored with the same color.

Again the elements of the set $A_{ij}, i=[\alpha_1/4], j=[\alpha_2/4]-1$are not adjacent among themselves but are adjacent to every vertex of the clique and also adjacent to all $B_{ij},E_j,i\geq[\alpha_1/4], j\geq[\alpha_2/4]$. The elements of $A_{ij}, i=[\alpha_1/4]-1, j=[\alpha_2/4]$are not adjacent among themselves but are adjacent to the complete graph of order $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}-1$and also adjacent to$B_{ij},E_j,i\geq[\alpha_1/4], j\geq[\alpha_2/4], A_{ij}, i=[\alpha_1/4]-1, j=[\alpha_2/4]$. Therefore two more colors are needed to color these vertices.

All non adjacent elements outside the clique can be colored with some colors used in the clique as they are not adjacent to every member of the clique. Thus we have a proper coloring on exactly $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}+2$ colors. Hence $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{p^{\alpha_2}q}\right)=p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}+2$.

(b) If$\alpha_1\not\equiv 1,2(\text{mod}\,4)$, then the elements of the set $B_{ij}, i\geq[\alpha_2/4]$ and the set $E_j, j\geq[\alpha_1/4]$ are not adjacent with each other but are adjacent to very member of the above mentioned complete graph of order $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}-1$. Therefore the elements of the sets A_{ij},C_j,D_j for all $i\geq[\alpha_1/4], j\geq[\alpha_2/4]$along with any one element either from the set $i\geq[\alpha_1/4], j\geq[\alpha_2/4]$ or from the set $E_j, j\geq[\alpha_2/4]$ will induce a clique of order $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}$.

Thus we must have

$$\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{p^{\alpha_2}q}\right)\geq p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}.$$

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Since all elements of the set A_{ij}, C_j, D_j for all $i \geq [\alpha_1/4], j \geq [\alpha_2/4]$ induce a complete subgraph of order $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}-1$, so these vertices can be colored with $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}-1$ colors. The elements of the set $B_{ij,} i \geq [\alpha_1/4], j \geq [\alpha_2/4]$ are adjacent to every member of the complete graph and the set $E_j, j \geq [\alpha_2/4]$ are adjacent to every member of the complete graph and so we can assign one more color to color these vertices. But all elements of B_{ij} and E_j are not adjacent to each other, so they can be colored with the same color.

All non adjacent elements outside the clique can be colored with some colors used in the clique as they are not adjacent to every member of the clique. Thus, we have a proper coloring on exactly $p^{[3\alpha_1/4]}p^{[3\alpha_2/4]}$ colors. Hence $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q}\right) \leq p^{[3\alpha_2/4]}p^{[3\alpha_2/4]}$. Thus $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{p^{\alpha_2}q}\right) = p^{[3\alpha_2/4]}p^{[3\alpha_2/4]}$.

Theorem 3.4 Let $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ be the nil graph of the commutative ring $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}$, where *p*, *q* and *r* are distinct primes and $\alpha_1, \alpha_2, \alpha_3$ be any positive integers such that at least two $\alpha_1 \equiv 1 or\, 2 (\mathrm{mod}\, 4)$, for all *i*=1,2,3 then $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)\right) = p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_1/4]}+1$.

Proof : The vertex set of the graph $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ can be partitioned into the following subsets

$A_{i,j,k} = \{(mp^i, m^/q^jr^k): p, q \nmid m, m^/, 1 \leq m \leq p^{\alpha_1-i}-1, 1 \leq m^/ \leq p^{\alpha_2-j}q^{\alpha_2-k} - 1\}, 0 \leq i \leq \alpha_1 - 1, 0 \leq j \leq \alpha_2 - 1, 0 \leq k \leq \alpha_3 - 1.$

$$B_i = \{(mp^i, 0): p, q \nmid m, 1 \leq m \leq p^{\alpha_1-i}-1\}, 0 \leq i \leq \alpha_1 - 1$$

$C_{j,k} = \{(0, m^/q^jr^k): q, r \nmid m^/, 1 \leq m^/ \leq q^{\alpha_2-j}r^{\alpha_3-k}-1\}, 0 \leq j \leq \alpha_2 - 1, 0 \leq k \leq \alpha_3 - 1.$

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$$D_{i,k} = \{(mp^i, m^/ r^k): p \nmid m; q, r \nmid m^/, 1 \le m \le p^{\alpha_1 - i} - 1, 1 \le m^/ \le r^{\alpha_3 - k} q^{\alpha_2} - 1\}, 0 \le i \le \alpha_1 - 1, 0 \le k \le \alpha_3$$

$$E_{i,j} = \{(mp^i, m^/ q^j): p \nmid m,, q \nmid m^/, 1 \le m \le p^{\alpha_1 - i} - 1, 1 \le m^/ \le q^{\alpha_2 - j} r^{\alpha_3} - 1\}, 0 \le i \le \alpha_1 - 1, 0 \le j \le \alpha_2.$$

$$F_j = \{(0, m^/ q^j): q, r \nmid m^/, 1 \le m^/ \le q^{\alpha_2 - j} r^{\alpha_3} - 1\}, 0 \le j \le \alpha_2$$

$$G_k = \{(0, m^/ r^k): q, r \nmid m^/, 1 \le m^/ \le q^{\alpha_2} r^{\alpha_3 - k} - 1\}, 0 \le k \le \alpha_3$$

Let (x_1, y_1) and $(x_2 y_2)$ be any two vertices of $\Gamma_N\left(\square_{p^{\alpha_1}} \times \square_{q^{\alpha_2} r^{\alpha_3}}\right)$.

Then $x_1 x_2 = m_1 p^{i_1} m_2 p^{i_2} = m_1 m_2 p^{i_1 + i_2}$ and

$y_1 y_2 = m_2^/ p^{i_1} q^{k_1} m_2^/ p^{j_2} q^{k_2} = m_1^/ m_2^/ p^{j_1 + j_2} q^{k_1 + k_2}$

Then $(x_1, y_1)(x_2 y_2) \in N\left(\square_{p^{\alpha_1}} \times \square_{q^{\alpha_2} r^{\alpha_3}}\right)$ if $i_1 + i_2 \ge [\alpha_1 / 2]$ and $j_1 + j_2 \ge [\alpha_2 / 2]$ and $k_1 + k_2 \ge [\alpha_2 / 2]$, i.e. if $(x_1 x_2)^2 \equiv 0(\bmod p^{\alpha_1})$ and $(y_1 y_2)^2 \equiv 0(\bmod q^{\alpha_2} r^{\alpha_3})$.

Now we can consider the following cases :

Case 1 : For the vertex set A_{ijk}, let $(x_1, y_1) \in A_{i_1 j_1 k_1}$ and $(x_2, y_2) \in A_{i_2 j_2 k_2}$ such that $x_1 = m_1 p^{i_1}, x_2 = m_2 p^{i_2}$ and $y_1 = m_1^/ q^{j_1} r^{k_1}, y_2 = m_2^/ q^{j_2} r^{k_2}$. Now we have the following subcases :

(i) For $0 \le i \le [\alpha_1 / 4] - 1, 0 \le j \le [\alpha_2 / 4] - 1, 0 \le k \le [\alpha_3 / 4] - 1$,

$(x_1, y_1)(x_2, y_2) \notin N\left(\square_{p^{\alpha_1}} \times \square_{q^{\alpha_2} r^{\alpha_3}}\right)$ as $i_1 + i_2 < [\alpha_1 / 2], j_1 + j_2 < [\alpha_2 / 2]$ and $k_1 + k_2 < [\alpha_2 / 2]$. So (x_1, y_1) and (x_2, y_2) are not adjacent. Therefore the elements of $A_{ijk}, i \le [\alpha_1 / 4] - 1, j \le [\alpha_1 / 4] - 1$ and $k \le [\alpha_3 / 4] - 1$ are not adjacent to each other. But these elements are adjacent to the elements of $A_{i^/ j^/ k^/}$ such that $i + i^/ \ge [\alpha_2 / 2]$ and $j + j^/ \ge [\alpha_2 / 2]$ and $k + k^/ \ge [\alpha_3 / 2]$ for some $i^/, j^/, k^/$.

(ii) For $[\alpha_1 / 4] \le i \le \alpha_1 - 1, [\alpha_2 / 4] \le j \le \alpha_2 - 1$ and

$k \le [\alpha_3 / 4] - 1, (x_1, y_1).(x_2 y_2) a_1 \in N\left(\square_{p^{\alpha_1}} \times \square_{p^{\alpha_2} q}\right)$ as $i_1 + i_2 \ge [\alpha_1 / 2], j_1 + j_2 \ge [\alpha_2 / 2]$ and $k_1 + k_2 \ge [\alpha_3 / 2]$. Therefore the elements of the set $A_{ijk}, i \ge (\alpha_1 / 4), j \ge [\alpha_2 / 4], k \ge [\alpha_3 / 4]$ are adjacent to each other and adjacent to the elements of

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$A_{i^{/} j^{/} k^{/}}, i^{/} \geq (\alpha_1/4), j^{/} \geq [\alpha_2/4], k^{/} \geq [\alpha_3/4]$. Total number of elements of $UA_{ijk}, i \geq (\alpha_1/4), j \geq [\alpha_2/4], k \geq [\alpha_3/4]$ is $\left(p^{[3\alpha_1/4]}-1\right)\left(q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}-1\right)$.

Case 2 : For the vertex set B_i, let $((x_1,0) \in B_{i_1}$ and $(x_2,0) \in B_{i_2}$, then we have the following subcases:

(i) For $0 \leq i_1, i_2 \leq [\alpha_2/4]-1$, the elements of the set B_{i_2} and B_{i_2} are not adjacent with each other since $(x_1x_2)^2 \not\equiv 0 \left(\bmod p^{\alpha_1}\right)$ as $i_1 + i_2 < [\alpha_2/2]$. Hence $(x_1x_2, 0) \notin N\left(\square p^{\alpha_1} \times \square_{q^{\alpha_2}r^{\alpha_3}}\right)$.

(ii) For $[\alpha_2/4] \leq i_2, i_2 \leq \alpha_1 - 1$, the elements of the set B_{i_1} and B_{i_2} are adjacent with each other since $(x_1x_2) \equiv 0 \left(\bmod p^{\alpha_1}\right)$ as $i_1 + i_2 \geq [\alpha_1/2]$ and hence $(x_1x_2, 0) \in N\left(\square_{p^{\alpha_1}} \times \square_{q^{\alpha_2}r^{\alpha_3}}\right)$. There are $\left(p^{[3\alpha_1/4]}-1\right)$ such elements in this set. All these elements are adjacent to all those of the set $A_{i,j,k} i \geq [\alpha_1/4], j \geq [\alpha_2/4], k \geq [\alpha_3/4]$

Case 3: For the vertex set C_{jk}, let $(0, y_1) \in C_{j_1k_1}$ and $C_{jk}, let (0, y_2) \in C_{j_2k_2}$ then we have the following subcases:

(i) For $0 \leq j_1, j_2 \leq [\alpha_2/4]-1, 0 \leq k_1, k_2 \leq [\alpha_3/4]-1$, the elements of the set $C_{j_1k_1}$ and $C_{j_2k_2}$ are not adjacent with each other since $(y_1y_2)^2 \not\equiv 0 \left(\bmod p^{\alpha_1}\right)$ as $j_1 + j_2 < [\alpha_2/2]$ and $k_1 + k_2 < [\alpha_3/2]$. Hence $(0, y_1, y_2) \notin N\left(\square_{p^{\alpha_1}} \times \square_{a^{\alpha_2}r^{\alpha_3}}\right)$.

(ii) For $[\alpha_2/4] \leq j_1, j_2 \leq \alpha_2 - 1, [\alpha_3/4] \leq k_1, k_2 \leq \alpha_3 - 1$, the elements of the set $C_{j_1k_1}$ and $C_{j_2k_2}$ are adjacent with each other since $(y_1y_2)^2 \equiv 0 \left(\bmod p^{\alpha_1}\right)$ and hence $(0, y_1y_2) \in N\left(\square_{p^{\alpha_1}} \times \square_{q^{\alpha_2}r^{\alpha_3}}\right)$. There are $\left(q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}-1\right)$ such elements in this set. All these elements are adjacent to the elements of the sets which are considered in subcase1 (ii) and subcase2(ii)

Case 4: For the vertex set $D_{i,k}$, let $(x_1, y_1), (x_2, y_2) \in D_{i,k}$.

Then for $0 \leq i \leq \alpha_1 - 1, 0 \leq k \leq \alpha_3, (x_1, y_1)(x_2y_2) \notin N\left(\square_{p^{\alpha_1}} \times \square_{q^{\alpha_2}r^{\alpha_3}}\right)$

as $(y_1y_2)^2 \not\equiv 0 \left(\bmod q^{\alpha_2}r^{\alpha_3}\right)$.

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These elements are adjacent to the elements of the sets $A_{i^{/},j^{/},k^{/}}, B_{i^{/}}, C_{j^{/}k^{/}}$ if $i+i^{/} \geq [\alpha_2/2], k+k^{/} \geq [\alpha_3/2]$ and $j^{/} \geq [\alpha_3/2]$.

Case 5 : For the vertex set E_{ij}, let $(x_1, y_1).(x_2, y_2) \in E_{ik}$. Then for $0 \leq i \leq \alpha_1 - 1, 0 \leq j \leq \alpha_2, (x_1, y_1).(x_2, y_2) \notin N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ as

$(y_1 y_2)^2 \not\equiv 0\left(\bmod q^{\alpha_2} r^{\alpha_3}\right)$. These elements are adjacent to the set $A_{i^{/}j^{/}k^{/}}, B_{i^{/}}, C_{j^{/}k^{/}}$ if $i+i^{/} \geq [\alpha_3/2], j+j^{/} \geq [\alpha_2/2]$.

Case 6 : For the vertex set F_j, let $(0, y_1) \in F_{j_1}$ and $(0, y_1) \in F_{j_2}$. Then for $0 \leq j_1, j_2 \leq \alpha_2 - 1, (0, y_1)(0, y_2) \notin N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ as $(y_1 y_2)^2 \not\equiv 0\left(\bmod q^{\alpha_2} r^{\alpha_3}\right)$.

Case 7 : For the vertex set $G_{k,}$ let $(0, y_1) \in G_{k_1}$ and $(0, y_2) \in G_{k_2}$. Then for $0 \leq k_1, k_2 \leq \alpha_3 - 1, (0, y_1)(0, y_2) \notin N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ as $(y_1 y_2)^2 \not\equiv 0\left(\bmod q^{\alpha_2} r^{\alpha_3}\right)$.

Now considering all cases discussed above, the elements of $A_{ijk} \geq [\alpha_1/4], j \geq [\alpha_2/4], k \geq [\alpha_3/4]$ along with the elements of $B_i, i \geq [\alpha_1/4], C_{jk}, j \geq [\alpha_2/4], k \geq [\alpha_3/4]$ will induce a complete subgraph of order

$$\left(p^{[3\alpha_1/4]} - 1\right)\left(q^{[3\alpha_2/4]} r^{[3\alpha_3/4]} - 1\right) + \left(p^{[3\alpha_1/4]} - 1\right) + \left(q^{\left[\frac{3\alpha_2}{4}\right]} r^{\left[\frac{3\alpha_3}{4}\right]} - 1\right) =$$

$$p^{[3\alpha_1/4]} q^{[3\alpha_2/4]} r^{[3\alpha_3/4]} - 1$$

Again the elements of the set $B_i, i = [\alpha_1/4]$ and $C_{jk}, j = [\alpha_2/4], k = [\alpha_3/4]$ are adjacent with each other and also adjacent to every vertex of the above mentioned complete graph and induce a clique of order $p^{[3\alpha_1/4]} q^{[3\alpha_2/4]} r^{[3\alpha_3/4]} + 1$.

All elements of the clique can be colored with $p^{[3\alpha_1/4]} q^{[3\alpha_2/4]} r^{[3\alpha_3/4]} + 1$ colors. The elements of the sets D_{jk}, E_{ij}, F_j, G_k are not adjacent with each other and also not adjacent to every vertex of the complete graph. All these non adjacent elements outside the clique can be colored with some colors used in the clique as they are not

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adjacent to every member of the clique. So we can color the vertices of the graph $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ with exactly $p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}+1$ colors. Therefore

$$\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right) \leq p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]} + 1.$$

Hence $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right) = p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]} + 1.$

Theorem 3.5 : Let $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ be the nil graph of the commutative ring $\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}$ where p, q and r are distinct primes and $\alpha_1, \alpha_2, \alpha_3$ be any positive such that exactly one $\alpha_1 \equiv 1 \; or \; 2(\mathrm{mod}\, 4)$, for all i=1,2,3, then $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)\right) = p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}$.

Proof : If $\alpha_i \equiv 1 or\, 2(\mathrm{mod}\, 4)$, for exactly one $\alpha_i, i = 1,2,3$ then by the proof of the previous theorem we can see that the elements of subcase1(ii), subcase2(ii) and subcase3(ii) will induce a complete subgraph of order $p^{[3\alpha_1/4]}q^{[3\alpha_2 4]}r^{[3\alpha_3/4]}-1$. Again the elements of the set $A_{ijk}, i = [\alpha_1/4]-1, j = [\alpha_2/4]-1, k = [\alpha_3/4]-1$ are not adjacent among themselves but are adjacent to all vertices of the complete graph and induce a clique of order $p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}$.

Thus $\chi(\Gamma_N(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}})) \geq p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}$.

All elements of the clique can be colored with $p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}$ colors. The elements of the set D_{jk}, E_{ij}, F_j, G_k are not adjacent with each other and also not adjacent to every vertex of the complete graph. All non adjacent elements outside the clique can be colored with some colors used in the clique as they are not adjacent to every member of the clique. Therefore all vertices of the graph can be properly colored with $p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}$ colors.

Thus $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right) \leq p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}$

Hence $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}} \times \mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right) = p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}$.

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Theorem 6.2.6 : Let $\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)$ be the nil graph of the commutative ring $\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}$, where *p*, *q* and r are distinct primes and $\alpha_1,\alpha_2,\alpha_3$ be any positive such that none of $\alpha_1 \equiv 1\ or\ 2 \pmod 4$, for all $i=1,2,3$ then $\chi\left(\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)\right)=p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}-1$.

Proof : If $\alpha_1 \not\equiv 1\ or\ 2 \pmod 4$, for $\alpha, i=1,2,3$ then by the proof of the previous theorem we can see that the elements considered in subcase1(ii), subcase2(ii) and subcase3(ii) will induce a clique of order $p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}-1$. $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)\geq p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3a_3/4]}-1$.

All elements of the clique can be colored with $p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3\alpha_3/4]}-1$ colors. The elements of the set D_{jk}, E_{ij}, F_j, G_k not adjacent with each other and also not adjacent to every vertex of the complete graph. All non adjacent elements outside the clique can be colored with some colors used in the clique as they are not adjacent to every member of the clique. Hence $\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)\leq p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3a_3/4]}-1$.

Hence

$$\chi\Gamma_N\left(\mathbb{Z}_{p^{\alpha_1}}\times\mathbb{Z}_{q^{\alpha_2}r^{\alpha_3}}\right)\geq p^{[3\alpha_1/4]}q^{[3\alpha_2/4]}r^{[3a_3/4]}-1$$

4. CONCLUSION

It has been concluded that the chromatic number of the nil graph $\Gamma_N\left(\mathbb{Z}_m\times\mathbb{Z}_n\right)$ depends on the prime factorization of *m* and *n*.

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