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Accelerated Homotopy Perturbation Method for Solution of Coupled Emden–Fowler Equations

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Abstract. We have implemented accelerated homotopy perturbation method (HPM) for the solution coupled Emden–Fowler equations which are nonlinear ordinary differential equation having singularity at $x = 0$. In this method, we have included a new form of He's polynomial for handling the nonlinear terms present in the equation. The result obtained from the proposed method shows the reliability, effectiveness and applicability of this method for solving such complex nonlinear coupled system.

Keywords: Accelerated He's polynomial, (Homotopy perturbation method) HPM, coupled Emden–Fowler equations.

1.Introduction

Lane-Emden equation represents the mathematical model of the physcial problem that occur in astrophysics and mathematical physics.

$$\frac{d^2u}{dx^2} + \frac{\beta_1}{x}\left(\frac{du}{dx}\right) + \kappa(u) = 0 \tag{*}$$

For the different values of parameter β_1 and function $\kappa(u)$, the above model [6] represents different phenomena used in theory of steller structure, thermal behaviour of spherical cloud of gas acting under the mutual enchantment of its molecules and concern to the classical laws of thermodynamics. The polytropic theory of stars genuinely follows out of thermodynamic considerations, that offers with the issue of energy transport, through the switch of fabric between exceptional levels of the star. These equations are one of the fundamental equations in the theory of stellar shape and has been the center of attention of many researchers. In 1995, Adomian [5] had applied decomposition method for the analytical solution of Lane-Emden equation. In 2003, Liao [14] proposed analytical solution of Lane-Emden equation by applying homotopy analysis method (HAM) and also discussed the significance of this method that the HAM method provides appropriate adjustment in the convergence region without using Pade approximation. In 2005, Wazwaz [12] used Adomian decomposition method (ADM) to provide the exact solution of generalized Emden–Fowler equation. Yildirim [10-11] used He's Homotopy perturbation method (HPM) [1,2] and variational iteration method [3] (VIM) to obtain the solution of singular initial value problem of Lane–Emden type equation. Tabrizidooz et. al. [9], used hybrid function method which consist of Lagrange's polynomial and block-pulse function to reduce the Emden–Fowler equations to a system of nonlinear equations. Dehghan [7], used variational iteration method (VIM) and computed the approximate solution of Lane-Emden equation. Further, Dehghan [8], had

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applied hermite collocation technique to solve nonlinear Lane–Emden type equations. In 2011, Wazwaz [13] used VIM with Langrange correction multiplier to solve the system of coupled Lane-Emden equations. Moreover, Muatjetjeja [15] studied the classification of generalized coupled Lane-Emden equation using First order Lagrangian. In this paper, we have applied accelerated HPM to obtain the series solution of nonlinear coupled Lane-Emden equations. Here, we have used a new form of He's polynomial introduced by Kalla [4] which accelerate the convergence of the HPTM.

2.Accelerated Homotopy Perturbation Method

To elucidate the proposed technique, consider the following nonlinear equations:

$$\frac{d^2u}{dx^2} + \frac{\beta_1}{x}\left(\frac{du}{dx}\right) + f_1(u,v) = k_1(x)$$
$$\frac{d^2v}{dx^2} + \frac{\beta_2}{x}\left(\frac{dv}{dx}\right) + f_2(u,v) = k_2(x) \tag{1}$$

with condition $u(0) = c_1$, $u'(0) = 0$, $v(0) = c_2$ and $v'(0) = 0$.
On solving eq. (1), we get

$$u(x) = u(0) + \int_0^x \frac{1}{s^{\beta_1}}\left(\int_0^s \xi^{\beta_1}\left(k_1(\xi) - f_1(u,v)\right)d\xi\right)ds$$

and

$$v(x) = v(0) + \int_0^x \frac{1}{s^{\beta_2}}\left(\int_0^s \xi^{\beta_2}\left(k_2(\xi) - f_2(u,v)\right)d\xi\right)ds \tag{2}$$

Now apply Homotopy perturbation method to solve eq.(2), i.e. assuming

$u(x) = \sum_{n=0}^{\infty} u_n p^n$ and $v(x) = \sum_{n=0}^{\infty} v_n p^n$

$$f_1(u,v) = L_1 f_1(u,v) + N_1 f_1(u,v)$$

and

$$f_2(u,v) = L_2 f_2(u,v) + N_2 f_2(u,v)$$

where Lf_1 and $L_1\ f_1$ are linear functions of u, v and Nf_1, Nf_2 the nonlinear functions of u, v.

Eq. (2) reduces to

$$\sum_{n=0}^{\infty} u_n p^n = c_1 + p\left(\int_0^x \frac{1}{s^{\beta_1}}\left(\int_0^s \xi^{\beta_1}\left(k_1(\xi) - L_1 f_1\left(\sum_{n=0}^{\infty} u_n p^n, \sum_{n=0}^{\infty} v_n p^n\right) - \sum_{n=0}^{\infty} \breve{H}_{1n} p^n\right)d\xi\right)ds\right)$$

and

$$\sum_{n=0}^{\infty} v_n p^n = c_2 + p\left(\int_0^x \frac{1}{s^{\beta_2}}\left(\int_0^s \xi^{\beta_2}\left(k_2(\xi) - L_2 f_2\left(\sum_{n=0}^{\infty} u_n p^n, \sum_{n=0}^{\infty} v_n p^n\right) - \sum_{n=0}^{\infty} \breve{H}_{2n} p^n\right)d\xi\right)ds\right) \tag{3}$$

where $p \in [0,1]$ is a parameter, $\breve{H}_{1n}$ and $\breve{H}_{2n}$ are accelerated He's polynomial

$$N_i f_i(u,v) = \sum_{n=0}^{\infty} \breve{H}_{in} p^n,$$
$$\text{and } \breve{H}_{in}(u,v) = N(S_{in}) - \sum_{j=0}^{n-1} \breve{H}_{ij},\ n \geq 1,\ i = 1,2 \tag{4}$$

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Where $\tilde{H}_{i0} = N_i(f_i(u_0, v_0)), S_{1k} = (u_0 + u_1 + \cdots + u_k)$ and $S_{2k} = (v_0 + v_1 + \cdots + v_k)$.
Now, comparing coefficients of the equal power of p, we have

$$p^0: u_0 = u(0) = c_1$$

$$p^1: u_1 = \int_0^x \frac{1}{s^{\beta_1}} \left(\int_0^s \xi^{\beta_1} \left(k_1(\xi) - L_1 f_1(u_0, v_0) - \tilde{H}_{10} \right) d\xi \right) ds$$

$$p^2: u_2 = -\int_0^x \frac{1}{s^{\beta_1}} \left(\int_0^s \xi^{\beta_1} \left(L_1 f_1(u_1, v_1) + \tilde{H}_{11} \right) d\xi \right) ds$$

$$\vdots$$

$$p^0: v_0 = v(0) = c_2$$

$$p^1: v_1 = \int_0^x \frac{1}{s^{\beta_2}} \left(\int_0^s \xi^{\beta_2} \left(k_2(\xi) - L_2 f_2(u_0, v_0) - \tilde{H}_{20} \right) d\xi \right) ds$$

$$p^2: v_2 = -\int_0^x \frac{1}{s^{\beta_2}} \left(\int_0^s \xi^{\beta_2} \left(L_2 f_2(u_1, v_1) + \tilde{H}_{21} \right) d\xi \right) ds$$

$$\vdots$$

Hence, the solution of eq.(1) is obtained by taking $p \to 1$, i.e.
$u(x) = \sum_{n=0}^{\infty} u_n$ and $v(x) = \sum_{n=0}^{\infty} v_n$

3.Condition of convergence of Accelerated HPTM

Here, we emphasize on condition of convergence of the above introduced method.

Theorem 3.1: If u, v, u_k and $v_k \in \boldsymbol{B}$, where $\boldsymbol{B}$ is Banach space, then the series

$$u(x) = \sum_{n=0}^{\infty} u_n p^n \text{ and } v(x) = \sum_{n=0}^{\infty} v_n p^n$$

converges to the solution of eq.(1) if $\|u_{n+1}\| \le \kappa \|u_n\|$ and $\|v_{n+1}\| \le \eta \|v_n\|$ where $0 < \kappa, \eta < 1$.This conditions of convergence of the series is proved in [5-6].

4.Application of Accelerated HPM

Example 1: Consider following nonlinear coupled Emden–Fowler equations

$$\frac{d^2u}{dx^2} + \frac{1}{x}\left(\frac{du}{dx}\right) + u^2 v - (4x^2 + 5)u = 0,$$
$$\frac{d^2v}{dx^2} + \frac{2}{x}\left(\frac{dv}{dx}\right) + uv^2 - (4x^2 - 5)v = 0$$

With initial conditions $u(0) = 1$, $u'(0) = 0$, $v(0) = 1$ and $v'(0) = 0$. (5)

The exact solution is given as $u(x) = e^{x^2}$ and $v(x) = e^{-x^2}$.

On applying accelerated Homotopy perturbation technique on eq.(5), we get

$$\sum_{n=0}^{\infty} u_n p^n = u(0) + p\left(\int_0^x \frac{1}{s} \left(\int_0^s \xi \left((4x^2 + 5) \left(\sum_{n=0}^{\infty} u_n p^n \right) - \sum_{n=0}^{\infty} \tilde{H}_{1n} p^n \right) d\xi \right) ds \right)$$

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and

$$\sum_{n=0}^{\infty} v_n p^n = v(0) + p\left(\int_0^x \frac{1}{s^2}\left(\int_0^s \xi^2\left((4x^2-5)\left(\sum_{n=0}^{\infty} v_n p^n\right) - \sum_{n=0}^{\infty} \widetilde{H}_{2n} p^n\right) d\xi\right) ds\right) \tag{6}$$

Where $u(x) = \sum_{n=0}^{\infty} u_n p^n$ and $v(x) = \sum_{n=0}^{\infty} v_n p^n$, $\sum_{n=0}^{\infty} \widetilde{H}_{1n} p^n = u^2 v$ and

$$\sum_{n=0}^{\infty} \widetilde{H}_{2n} p^n = uv^2$$

A few components of accelerated He's polynomial are given as

$$\widetilde{H}_{10} = u_0^2 v_0$$

$$\widetilde{H}_{11} = 2u_0 u_1 v_0 + u_1^2 v_0 + u_0^2 v_1 + 2u_0 u_1 v_1 + u_1^2 v_1$$

$$\widetilde{H}_{12} = 2u_0 u_2 v_0 + 2u_1 u_2 v_0 + u_2^2 v_0 + 2u_0 u_2 v_1 + 2u_1 u_2 v_1 + u_2^2 v_1 + u_0^2 v_2 + 2u_0 u_1 v_2 + u_1^2 v_2 + 2u_0 u_2 v_2 + 2u_1 u_2 v_2 + u_2^2 v_2$$

$$\vdots$$

$$\widetilde{H}_{20} = u_0 v_0^2$$

$$\widetilde{H}_{21} = u_1 v_0^2 + 2u_0 v_0 v_1 + 2u_1 v_0 v_1 + u_0 v_1^2 + u_1 v_1^2$$

$$\widetilde{H}_{22} = u_2 v_0^2 + 2u_2 v_0 v_1 + u_2 v_1^2 + 2u_0 v_0 v_2 + 2u_1 v_0 v_2 + 2u_2 v_0 v_2 + 2u_0 v_1 v_2 + 2u_1 v_1 v_2 + 2u_2 v_1 v_2 + u_0 v_2^2 + u_1 v_2^2 + u_2 v_2^2$$

$$\vdots \tag{7}$$

On equating the term containing equal power of p on the both sides of the eq. (6), we get

$$p^0: u_0 = u(0) = 1$$

$$p^1: u_1 = x^2 + \frac{x^4}{4}$$

$$p^2: u_2 = \frac{x^4}{4} + \frac{37x^6}{240} + \frac{x^8}{40} + \frac{11x^{10}}{8000} - \frac{x^{12}}{3840} - \frac{x^{14}}{15680}$$

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$$p^3: u_3 = \frac{x^6}{80} + \frac{109x^8}{7168} + \frac{73x^{10}}{14000} + \frac{38671x^{12}}{53222400} + \frac{634639x^{14}}{28252224000} - \frac{2544571x^{16}}{516612096000} + \frac{403999x^{18}}{81729648000} + \frac{2420057x^{20}}{1153152000000} + \frac{3302941931x^{22}}{1025555731200000} - \frac{1867463x^{24}}{166905446400000} - \frac{905977057x^{26}}{6111521095680000} - \frac{14644272829x^{28}}{531593846784000000} - \frac{2113860611x^{30}}{284782417920000000} + \frac{58080058721x^{32}}{1080063688704000000} + \frac{40645401061x^{34}}{43894463348736000000} + \frac{879057841x^{36}}{31895630807040000000} - \frac{6942109x^{38}}{609223424409600000000} - \frac{6936343x^{40}}{607535824896000000000} + \frac{x^{42}}{1014860980224000000} + \frac{x^{44}}{99957817344000000}$$

$$\vdots$$

$$p^0: v_0 = v(0) = 1,$$

$$p^1: v_1 = -x^2 + \frac{x^4}{5},$$

$$p^2: v_2 = \frac{3x^4}{10} - \frac{31x^6}{280} + \frac{x^8}{240} + \frac{x^{10}}{11000} + \frac{x^{12}}{2600} - \frac{x^{14}}{21000},$$

$$p^3: v_3 = -\frac{47x^6}{840} + \frac{557x^8}{17280} - \frac{3769x^{10}}{924000} - \frac{14171x^{12}}{41184000} + \frac{7888871x^{14}}{141261120000} - \frac{33973x^{16}}{12475008000} + \frac{359593x^{18}}{132723360000} + \frac{3697x^{20}}{847566720000} - \frac{152469049x^{22}}{10483458585600000} - \frac{1077933397x^{24}}{279697017600000000} + \frac{14825663393x^{26}}{113445110338560000000} + \frac{17360347063x^{28}}{11481873346944000000000} - \frac{2544556 5773x^{30}}{1803486369484800000000} - \frac{55991557x^{32}}{29382312960000000000} - \frac{225993583x^{34}}{801280159680000000000} + \frac{49502371579x^{36}}{5650439149555200000000000} + \frac{334133x^{38}}{5443073475840000000000} - \frac{115537x^{40}}{50194784640000000000000} - \frac{269x^{42}}{2783100902400000000} + \frac{x^{44}}{1369146240000000000}$$

$$\vdots$$

The solution of eq.(5) is given by $u(x) = \sum_{n=0}^{\infty} u_n p^n$ and $v(x) = \sum_{n=0}^{\infty} v_n p^n$ as $p \to 1$ i.e.

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$$u(x) = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{1441x^8}{35840} + \frac{369x^{10}}{56000} + \frac{24811x^{12}}{53222400} - \frac{1167161x^{14}}{28252224000} - \frac{2544571x^{16}}{516612096000} + \frac{403999x^{18}}{81729648000} + \frac{2420057x^{20}}{1153152000000} + \frac{3302941931x^{22}}{10255557312000000} - \frac{1867463x^{24}}{166905446400000} - \frac{905977057x^{26}}{611152109568000000} - \frac{14644272829x^{28}}{5315938467840000000} - \frac{2113860611x^{30}}{2847824179200000000} + \frac{5808005872 1x^{32}}{1080063688704000000000} + \frac{40645401061x^{34}}{4389446334873600000000} + \frac{879057841x^{36}}{3189563080704000000000} - \frac{6942109x^{38}}{609223424409600000000} - \frac{6936343x^{40}}{6075358248960000000000} + \frac{x^{42}}{10148609802240000 0} + \frac{x^{44}}{99957817344 00000} \cdots$$

and

$$v(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{629x^8}{17280} - \frac{67x^{10}}{16800} + \frac{1669x^{12}}{41184000} + \frac{1162151x^{14}}{14126112000} - \frac{33973x^{16}}{12475008000} + \frac{359593x^{18}}{132723360000} + \frac{3697x^{20}}{847566720000} - \frac{152469049x^{22}}{10483458585600000} - \frac{1077933397x^{24}}{279697017600000000} + \frac{14825663393x^{26}}{1134451103385600000 0} + \frac{17360347063x^{28}}{1148187334694400000 0} - \frac{25445565773x^{30}}{1803486369484800000 00} - \frac{55991557x^{32}}{29382312960000000 00} - \frac{225993583x^{34}}{80128015968000000 00} + \frac{49502371579x^{36}}{565043914955520000 00000} + \frac{334133x^{38}}{54430734758400000 000} - \frac{115537x^{40}}{50194784640000000 000} - \frac{269x^{42}}{27831009024000000 0} + \frac{x^{44}}{136914624000000 00} + \cdots$$

Table 1: Tabulated value of $u(x)$ and $v(x)$ up to fourth order approximation using Acc. HPM

x	$Approx.u(x)$	$Exact\ u(x)$	$Approx.v(x)$	$Exact\ v(x)$
0.1	1.01005016708416 9	1.01005016708416 8	0.99004983374920 25	0.99004983374916 81
0.2	1.04081077419315 07	1.04081077419238 82	0.96078943918663 74	0.96078943915232 32
0.3	1.09417428373781 37	1.09417428370521 04	0.91393118715516 81	0.91393118527122 82
0.4	1.17351087125521 42	1.17351087099181 03	0.85214382017174 53	0.85214378896621 13
0.5	1.28402541478545 66	1.28402541668774 14	0.77880104841449 84	0.77880078307140 49
0.6	1.43332936375237 78	1.43332941456034 01	0.69767779245252 1	0.69767632607103 1
0.7	1.63231572150438 82	1.63231621995537 91	0.61263236125789 98	0.61262639418441 6
0.8	1.89647756979253 93	1.89648087930495 17	0.52731166315345 3	0.52729242404304 85
0.9	2.24789070067186 7	2.24790798667647 17	0.44490951858242 69	0.44485806622294 11
1.0	2.71820556776560 94	2.71828182845904 5	0.36799697992723 523	0.36787944117144 233
1.1	3.35318833848828 35	3.35348465254902 45	0.29843090736319 4	0.29819727942988 733
1.2	4.21965365486734 7	4.22069581699655 4	0.23733591379990 915	0.23692775868212 165
1.3	5.41609932214327 5	5.41948070513120 95	0.18514773399090 736	0.18451952399298 915
1.4	7.08906761257348 1	7.09932706515663 5	0.14170424830383 258	0.14085842092104 495
1.5	9.45834173477887 9	9.48773583635853 3	0.10637272467794 33	0.10539922456186 425

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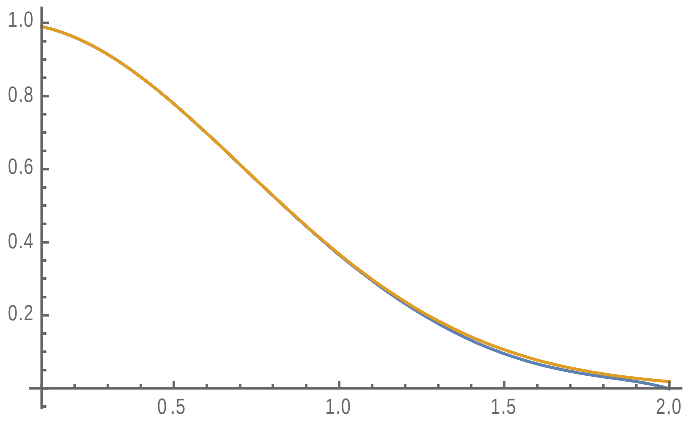

Fig.1: Plot of computed value of $v(x)$ and exact vaue of $v(x)$, when $0.1 \leq x \leq 2$.

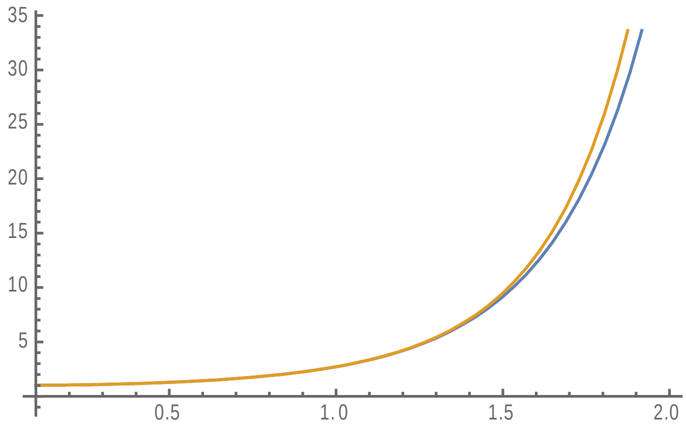

Fig.2: Plot of computed value of $u(x)$ and exact vaue of $u(x)$, when $0.1 \leq x \leq 2$.

Example 2: Consider following nonlinear coupled Emden–Fowler equations

$$\frac{d^2u}{dx^2} + \frac{2}{x}\left(\frac{du}{dx}\right) + v^2 - u^2 + 6v = 6 + 6x^2,$$
$$\frac{d^2v}{dx^2} + \frac{2}{x}\left(\frac{dv}{dx}\right) + u^2 - v^2 - 6v = 6 - 6x^2$$

With initial conditions $u(0) = 1$, $u'(0) = 0$, $v(0) = -1$ and $v'(0) = 0$. (8)

The exact solution is given as $u(x) = x^2 + e^{x^2}$ and $v(x) = x^2 - e^{x^2}$.

On applying accelerated Homotopy perturbation technique on eq.(8), we get

$$\sum_{n=0}^{\infty} u_n p^n = u(0) + p\left(\int_0^x \frac{1}{s^2}\left(\int_0^s \xi^2\left(\left(4\xi^2 + 6\right) - 6\left(\sum_{n=0}^{\infty} v_n p^n\right) + \sum_{n=0}^{\infty} \widetilde{H}_{1n}p^n\right)d\xi\right) ds\right)$$

and

$$\sum_{n=0}^{\infty} v_n p^n = v(0) + p\left(\int_0^x \frac{1}{s^2}\left(\int_0^s \xi^2\left(\left(6 - 6\xi^2\right) + 6\left(\sum_{n=0}^{\infty} v_n p^n\right) + \sum_{n=0}^{\infty} \widetilde{H}_{2n}p^n\right)d\xi\right) ds\right)$$

(9)

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Where $u(x) = \sum_{n=0}^{\infty} u_n p^n$ and $v(x) = \sum_{n=0}^{\infty} v_n p^n$, $\sum_{n=0}^{\infty} \widetilde{H}_{1n} p^n = u^2 - v^2$ and

$$\sum_{n=0}^{\infty} \widetilde{H}_{2n} p^n = v^2 - u^2$$

A few components of accelerated He's polynomial are given as

$$\widetilde{H}_{10} = u_0^2 - v_0^2$$
$$\widetilde{H}_{11} = 2u_0u_1 + u_1^2 - 2v_0v_1 - v_1^2$$
$$\widetilde{H}_{12} = 2u_0u_2 + 2u_1u_2 + u_2^2 - 2v_0v_2 - 2v_1v_2 - v_2^2$$
$$\vdots$$

$$\widetilde{H}_{20} = v_0^2 - u_0^2$$
$$\widetilde{H}_{21} = -2u_0u_1 - u_1^2 + 2v_0v_1 + v_1^2$$
$$\widetilde{H}_{22} = -2u_0u_2 - 2u_1u_2 - u_2^2 + 2v_0v_2 + 2v_1v_2 + v_2^2$$
$$\vdots$$

(10)

On equating the term containing equal power of p on the both sides of the eq. (9) and using eq.(10), we get

$$p^0: u_0 = u(0) = 1,$$
$$p^1: u_1 = 2x^2 + \frac{3x^4}{10},$$
$$p^2: u_2 = \frac{x^4}{5} + \frac{29x^6}{210} + \frac{x^8}{60},$$
$$p^3: u_3 = \frac{x^6}{35} + \frac{19x^8}{840} + \frac{727x^{10}}{115500} + \frac{41x^{12}}{163800} - \frac{1829x^{14}}{9261000} - \frac{29x^{16}}{856800} - \frac{x^{18}}{615600},$$
$$p^4: u_4 = \frac{x^8}{420} + \frac{x^{10}}{440} + \frac{967x^{12}}{1126125} + \frac{54449x^{14}}{490490000} - \frac{13517851x^{16}}{600359760000} - \frac{193357883x^{18}}{192490348050000} - \frac{991490807x^{20}}{898288290900000} + \frac{537921061x^{22}}{8264252276280000} + \frac{933673x^{24}}{35895636000000} + \frac{6931x^{26}}{3085559568000} + \frac{x^{28}}{14996016000},$$
$$\vdots$$
$$p^0: v_0 = v(0) = -1,$$
$$p^1: v_1 = -\frac{3x^4}{10},$$
$$p^2: v_2 = -\frac{x^4}{5} - \frac{29x^6}{210} - \frac{x^8}{60},$$

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$$p^3: v_3 = -\frac{x^6}{35} - \frac{19x^8}{840} - \frac{727x^{10}}{115500} - \frac{41x^{12}}{163800} + \frac{1829x^{14}}{9261000} + \frac{29x^{16}}{856800} + \frac{x^{18}}{615600},$$

$$p^4: v_4 = -\frac{x^8}{420} - \frac{x^{10}}{440} - \frac{967x^{12}}{1126125} - \frac{54449x^{14}}{490490000} + \frac{13517851x^{16}}{600359760000} + \frac{193357883x^{18}}{19249034805000} + \frac{991490807x^{20}}{898288290900000} - \frac{537921061x^{22}}{826425227628000} - \frac{933673x^{24}}{35895636000000} - \frac{6931x^{26}}{3085559568000} - \frac{x^{28}}{14996016000},$$

$$\vdots$$

The solution of eq.(8) is given by $u(x) = \sum_{n=0}^{\infty} u_n p^n$ and $v(x) = \sum_{n=0}^{\infty} v_n p^n$ as $p \to 1$ i.e.

$$u(x) = 1 + 2x^2 + \frac{x^4}{2} + \frac{x^6}{6} + \frac{x^8}{24} + \frac{1979x^{10}}{231000} + \frac{9991x^{12}}{9009000} - \frac{163621x^{14}}{1891890000} - \frac{33838151x^{16}}{600359760000} - \frac{449253241x^{18}}{38498069610000} - \frac{991490807x^{20}}{898288290900000} + \frac{537921061x^{22}}{826425227628000} + \frac{933673x^{24}}{35895636000000} + \frac{6931x^{26}}{3085559568000} + \frac{x^{28}}{14996016000} + \cdots$$

$$v(x) = -1 - \frac{x^4}{2} - \frac{x^6}{6} - \frac{x^8}{24} - \frac{1979x^{10}}{231000} - \frac{9991x^{12}}{9009000} + \frac{163621x^{14}}{1891890000} + \frac{33838151x^{16}}{600359760000} + \frac{449253241x^{18}}{38498069610000} + \frac{991490807x^{20}}{898288290900000} - \frac{537921061x^{22}}{826425227628000} - \frac{933673x^{24}}{35895636000000} - \frac{6931x^{26}}{3085559568000} - \frac{x^{28}}{14996016000} + \cdots$$

Table 2: Tabulated value of $u(x)$ and $v(x)$ up to fourth order approximation using Acc. HPM

x	$Approx.u(x)$	$Exact\ u(x)$	$Approx.v(x)$	$Exact\ v(x)$
0.1	1.020050167084191	1.020050167084168	−1.000050167084191	−1.000050167084168
0.2	1.080810774215132 3	1.0808107741923882	−1.000810774215132	−1.000810774192388
0.3	1.1841742849228518	1.1841742837052105	−1.004174284922851	−1.004174283705210
0.4	1.3335108900076216	1.3335108709918102	−1.013510890007621	−1.013510870991810
0.5	1.5340255579597342	1.5340254166877414	−1.034025557959734	−1.034025416687741
0.6	1.7933299711324258	1.7933294145603402	−1.073329971132425	−1.07332941456034
0.7	2.122316722729932	2.1223162199553793	−1.142316722729931	−1.142316219955379
0.8	2.536471656202916	2.5364808793049516	−1.256471656202916	−1.256480879304951
0.9	3.0578279052253325	3.057907986676472	−1.437827905225332	−1.437907986676471
1.0	3.7178539065719143	3.718281828459045	−1.717853906571914	−1.718281828459045
1.1	4.5616688320243775	4.5634846525490245	−2.141668832024377	−2.143484652549024
1.2	5.65405670119511	5.660695816996554	−2.774056701195109	−2.780695816996553
1.3	7.087707702550315	7.10948070513121	−3.707707702550313	−3.729480705131209
1.4	8.993730251235142	9.059327065156635	−5.073730251235140	−5.139327065156635
1.5	11.553299541126657	11.737735836358535	−7.053299541126653	−7.237735836358532

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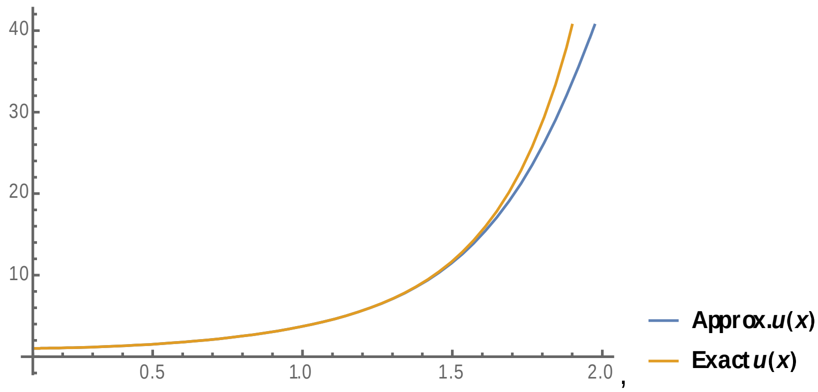

Fig.3: Plot of computed value of $u(x)$ and exact vaue of $u(x)$, when $0.1 \leq x \leq 2$

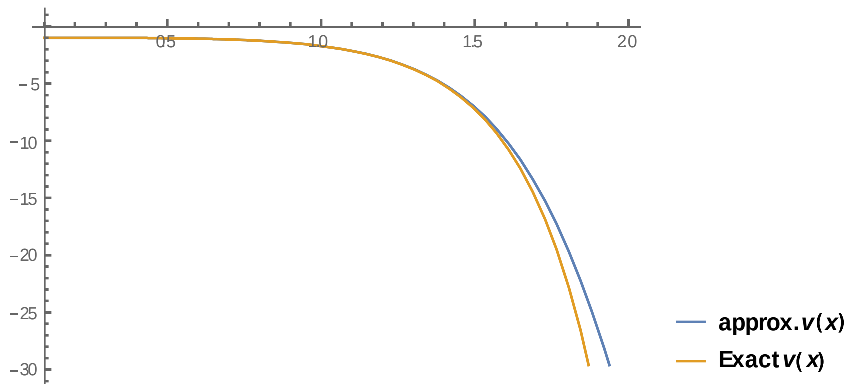

Fig.4: Plot of computed value of $v(x)$ and exact vaue of $v(x)$, when $0.1 \leq x \leq 2$

Example 3: Consider following nonlinear coupled Emden–Fowler equations

$$\frac{d^2u}{dx^2} + \frac{3}{x}\left(\frac{du}{dx}\right) + uv + (x^2 + 4)u^5 = 1,$$

$$\frac{d^2v}{dx^2} + \frac{4}{x}\left(\frac{dv}{dx}\right) + uv - (4x^2 + 5)u^3 = 1$$

With initial conditions $u(0) = 1$, $u'(0) = 0$, $v(0) = 1$ and $v'(0) = 0$. (11)

The exact solution is given as $u(x) = \frac{1}{\sqrt{1+x^2}}$ and $v(x) = \sqrt{1 + x^2}$

On applying accelerated Homotopy perturbation technique on eq.(11), we get

$$\sum_{n=0}^{\infty} u_n p^n = u(0) + p\left(\int_0^x \frac{1}{s^3}\left(\int_0^s \xi^3\left(1 - \sum_{n=0}^{\infty} \widetilde{H}_{1n} p^n\right) d\xi\right) ds\right)$$

and

$$\sum_{n=0}^{\infty} v_n p^n = v(0) + p\left(\int_0^x \frac{1}{s^4}\left(\int_0^s \xi^4\left(1 - \sum_{n=0}^{\infty} \widetilde{H}_{2n} p^n\right) d\xi\right) ds\right)$$

(12)

Where $u(x) = \sum_{n=0}^{\infty} u_n p^n$ and $v(x) = \sum_{n=0}^{\infty} v_n p^n$, $\sum_{n=0}^{\infty} \widetilde{H}_{1n} p^n = uv + (x^2 + 4)u^5$ and

$$\sum_{n=0}^{\infty} \widetilde{H}_{2n} p^n = uv - (4x^2 + 5)u^3$$

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A few components of accelerated He's polynomial are given as

$$\tilde{H}_{10} = u_0 v_0 + u_0^5(4 + x^2)$$

$$\tilde{H}_{11} = (5u_0^4 u_1 + 10u_0^3 u_1^2 + 10u_0^2 u_1^3 + u_1^5 + 5u_1^4 u_0)(4 + x^2) + u_1(v_0 + v_1) + u_0 v_1$$

$$\tilde{H}_{12} = (5u_0^4 u_2 + 5u_1^4 u_2 + 10u_1^3 u_2^2 + 10u_1^2 u_2^3 + 10u_0^3 u_2(2u_1 + u_2) + 10u_0^2 u_2(3u_1^2 + 3u_1 u_2 + u_2^2) + u_2^5 + 5u_2^4 u_1 + 20u_1^3 u_2 u_0 + 20u_1 u_2^3 + 5u_2^4 + +30u_1^2 u_2^2)(4 + x^2) + u_2(v_0 + v_1 + v_2) + (u_1 + u_0)v_2$$

$$\vdots$$

$$\tilde{H}_{20} = -u_0 v_0 + u_0^3(5 + 4x^2)$$

$$\tilde{H}_{21} = -u_1 v_0 - u_0 v_1 - u_1 v_1 + (3u_0^2 u_1 + 3u_0 u_1^2 + u_1^3)(5 + 4x^2)$$

$$\tilde{H}_{12} = -u_2 v_0 - u_2 v_1 - u_1 v_2 - u_2 v_2 - u_0 v_2 + (3u_0^2 u_2 + 3u_1^2 u_2 + 3u_1 u_2^2 + u_2^3 + 6u_0 u_1 u_2 + 3u_2^2)(5 + 4x^2)$$

$$\vdots$$

(13)

On equating the term containing equal power of p on the both sides of the eq. (12) and using eq.(13), we get

$$p^0: u_0 = u(0) = 1,$$

$$p^1: u_1 = -\frac{x^2}{2} - \frac{x^4}{24},$$

$$p^2: u_2 = \frac{5x^4}{12} - \frac{365x^6}{2688} + \frac{127x^8}{8960} + \frac{97x^{10}}{15120} - \frac{59x^{12}}{48384} - \frac{19x^{14}}{96768} + \frac{5x^{16}}{221184} + \frac{35x^{18}}{5971968} + \frac{13x^{20}}{29196288} + \frac{x^{22}}{65691648} + \frac{x^{24}}{4968677376},$$

$$p^3: u_3 = -\frac{475x^6}{2688} + \frac{1244179x^8}{5806080} - \frac{37691503x^{10}}{239500800} + \frac{299440093x^{12}}{3321077760} - \frac{2811931477x^{14}}{61024803840} + \frac{97525159097x^{16}}{4645713346560} - \frac{2393366801937773x^{18}}{291948240981196800} + \frac{263281173101810453x^{20}}{999111758024540160000} - \frac{30399835814649238 9x^{22}}{47957364385177927680 0} + \frac{67783083405649x^{24}}{9253369009384980 48} + \frac{19295933447856778 6223x^{26}}{79982420369520017276928 00} - \frac{73984530664263148399x^{28}}{4101662583052308578304000} + \frac{26969875195811069332 57x^{30}}{46407382368248977057382400 00} - \frac{54737214535604310970 97x^{32}}{620067761608042472401797120 0} + \cdots$$

$$\vdots$$

$$p^0: v_0 = v(0) = 1,$$

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$$p^1: v_1 = \frac{x^2}{2} + \frac{x^4}{7},$$

$$p^2: v_2 = -\frac{15x^4}{56} - \frac{229x^6}{4536} + \frac{871x^8}{29568} - \frac{167x^{10}}{174720} - \frac{x^{12}}{1536} - \frac{149x^{14}}{3290112} - \frac{x^{16}}{1050624},$$

$$p^3: v_3 = \frac{1025x^6}{9072} - \frac{124933x^8}{3193344} - \frac{5542417x^{10}}{1037836800} + \frac{30156221x^{12}}{3558297600} - \frac{163383869513x^{14}}{33197493288960} + \frac{1071489406963x^{16}}{480573236183040} - \frac{88329105197068 7x^{18}}{1226182612121026560} + \frac{263571040033695 11x^{20}}{16712414861501399040 0} - \frac{51229378170536 33x^{22}}{399644703209816064000} - \frac{30406824597563207x^{24}}{4708541230544741990400} + \frac{1224373435747028 81x^{26}}{42264762950365898342400} - \frac{31966710732179101x^{28}}{946068006507600936960 00} - \frac{16043258477651573771x^{30}}{17092004666877413425152000 0} + \frac{7947204249578830 7x^{32}}{25442643842241763737600 00} + \cdots$$

$$\vdots$$

The solution of eq.(11) is given by $u(x) = \sum_{n=0}^{\infty} u_n p^n$ and $v(x) = \sum_{n=0}^{\infty} v_n p^n$ as $p \to 1$ i.e.

$$u(x) = 1 - \frac{x^2}{2} + \frac{3x^4}{8} - \frac{5x^6}{16} + \frac{265295x^8}{1161216} - \frac{36155023x^{10}}{239500800} + \frac{42198619x^{12}}{474439680} - \frac{2823913447x^{14}}{61024803840} + \frac{976301782 97x^{16}}{464571334 6560} - \frac{23916557766217 73x^{18}}{29194824098119680 0} + \frac{26332565976002645 3x^{20}}{99911175802454016000} - \frac{30399105777181078 9x^{22}}{47957364385177927680 0} + \frac{6778326963969 7x^{24}}{92533690093849 8048} + \frac{192959334478567786223x^{26}}{79982420369520017276928 00} - \frac{73984530664263148399x^{28}}{410166258305230857830 4000} + \frac{2696987519581106933257x^{30}}{46407382368248977057382400 0} - \frac{5473721453560431097097x^{32}}{6200677616080424724017971200} - \frac{97169063530484074580 53x^{34}}{80326960026496411197505536000} + \cdots$$

$$v(x) = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{30865x^8}{3193344} - \frac{6534397x^{10}}{1037836800} + \frac{27839621x^{12}}{3558297600} - \frac{164887291433x^{14}}{33197493288960} + \frac{563701047 37x^{16}}{25293328220160} - \frac{883291051970687x^{18}}{1226182612121026560} + \frac{263571040033695 11x^{20}}{1671241486150139904 00} - \frac{512293781705363 3x^{22}}{399644703209816064000} - \frac{304068245975632 07x^{24}}{47085412305447419904 00} + \frac{1224373435747028 81x^{26}}{422647629503658983424 00} - \frac{31966710732179101x^{28}}{94606800650760093696000} - \frac{16043258477651573771x^{30}}{17092004666877413425152 0000} + \frac{79472042495788307x^{32}}{2544264384224176373760000} + \cdots$$

Table 3: Tabulated value of $u(x)$ and $v(x)$ up to third order approximation using Acc. HPM

x	$Approx.u(x)$	$Exact\ u(x)$	$Approx.v(x)$	$Exact\ v(x)$
0.1	0.9950371897696234	0.9950371902099893	1.0049875624027238	1.004987562112089
0.2	0.9805805697640929	0.9805806756909201	1.0198039746430532	1.019803902718552
0.3	0.9578238307029411	0.9578262852211513	1.0440323951034407	1.044030650891055
0.4	0.9284552727851157	0.9284766908852592	1.0770491243104938	1.0770329614269014
0.5	0.8943188819199225	0.8944271909999159	1.11812179673751	1.118033988749895
0.6	0.857107039828626	0.8574929257125443	1.1665292873255368	1.16619037896906
0.7	0.8181540104297477	0.8192319205190405	1.2216864784263035	1.2206555615733703
0.8	0.7783475995423381	0.7808688094430303	1.2832570282500424	1.2806248474865698
0.9	0.7381350682977187	0.7432941462471663	1.3512424249748631	1.3453624047073711
1.0	0.6975814470716805	0.7071067811865475	1.4260446388864347	1.4142135623730951

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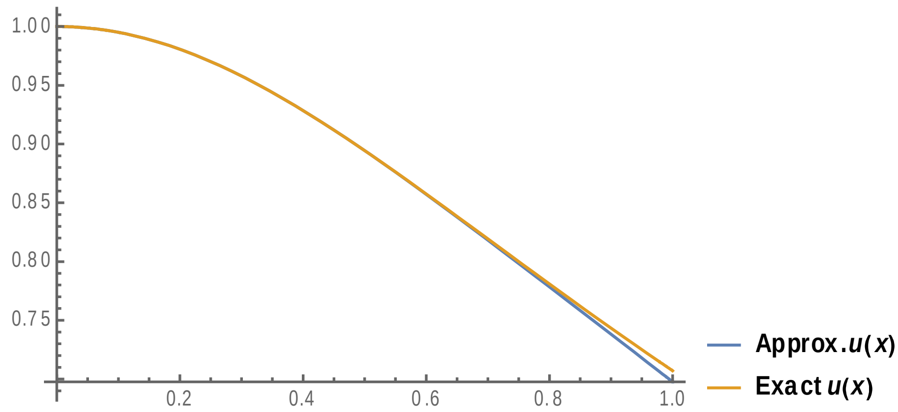

Fig.5: Plot of computed value of $u(x)$ and exact vaue of $u(x)$, when $0.1 \leq x \leq 1$

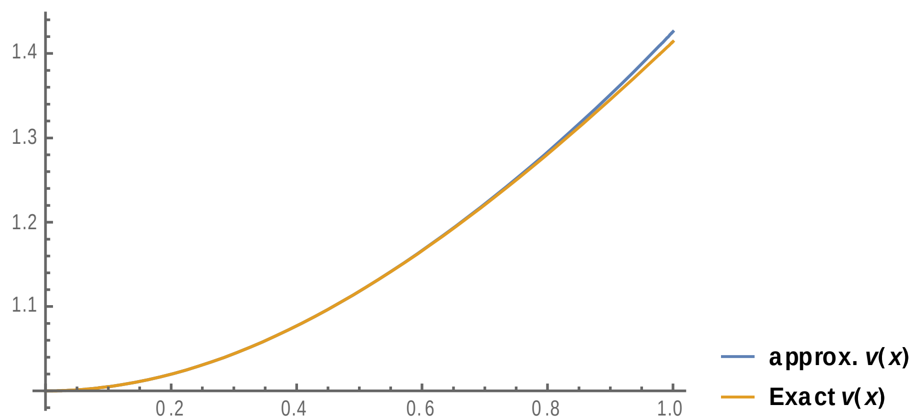

Fig.6: Plot of computed value of $v(x)$ and exact vaue of $v(x)$, when $0.1 \leq x \leq 1$

5. Conclusion

In this paper, we have applied accelerated HPM for the series solution of Coupled Emden–Fowler Equations. In these equations, the nonlinear term present in the coupled systems are handled with accelerated He's polynomial. In table 1, table 2 and table 3, we have compared the approximate solution obtained from accelerated HPM upto fourth and third order with the exact solution. We have used mathematica software 11.1 for plotting the graphs and to calculate the tabulated values. Fig. 1-6 represents the graph obtained from the computed value of $u(x)$ and $v(x)$ and the exact value of $u(x)$ and $v(x)$ for different values of x. The solution obtained from the accelerated homotpoy perturbation technique is close to the exact solution. Hence, we conclude that the accelerated HPM technique is efficient and reliable technique for solving nonlinear coupled system of equations.

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